

DISCRETE LOCALITIES I

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Introduction

Finite groups, Lie groups, and linear algebraic groups have various structural properties in common, but these properties tend to be deduced in different ways. For example, in finite group theory one has the notion of the generalized Fitting subgroup $F^*(G) = E(G)F(G)$ of a group G , where $E(G)$ plays much the same role for finite groups as reductive Lie groups or reductive algebraic groups play in their respective categories. The structure of centralizers of elements in simple groups belonging to any of these three classes reflects a similar convergence after the fact.

There remains some mystery as to why this should be so, even though most of the finite simple groups - as we now know - are in fact derivable from algebraic groups or from Lie groups by the methods pioneered by Chevalley. The alternating groups can be construed as being degenerate cases of algebraic or Lie groups via the theory of buildings, so one is left with only a small number (twenty-six) of exceptional cases. There are various strategies, of various degrees of complacency, for explaining these exceptions away, usually to the effect that anomalies are to be expected when working over very small finite fields. Such is the state of affairs that the entire enterprise of classifying the finite simple groups may be viewed as a vindication of the outlook which places Lie groups and algebraic groups first and foremost. Complacency becomes more difficult, however, when one considers not only the sporadic groups but also the “exotic fusion systems” which have been proliferating over the last fifteen years - and for which there are as yet no general organizing principles.

The aim of this series of papers (of which three are currently envisioned) is to begin at the other end of the thread, and to present a unified approach, from the view-point of a single prime p , to the three classes of groups already mentioned, to exotic fusion systems on finite p -groups, to the p -compact “groups” of Dwyer and Wilkerson, and to other structures whose investigation it will be an aim of these papers to initiate. The main objects of study will not be groups at all, but rather “partial groups” which may perhaps be thought of as “locally grouped spaces” in much the same way that, in algebraic

geometry, schemes are locally ringed spaces. The “affine spaces” in this analogy will be countable, locally finite groups G having the property that a certain lattice $\Omega_S(G)$ of subgroups of a maximal p -subgroup S of G is “finite-dimensional”.

For example, one may think of a linear algebraic group G over the algebraic closure of the field of p elements, and of a maximal unipotent subgroup S of G . In this example $\Omega_S(G)$ will be the poset (via inclusion) of subgroups of S which can be formed as the intersection of a set of G -conjugates of S . Using the Chevalley commutator formula, one finds that the members of $\Omega_S(G)$ are closed connected subgroups of S , and the dimension of S as a variety then provides an upper bound for the lengths of monotone chains in $\Omega_S(G)$. It is in this sense that $\Omega_S(G)$ is finite-dimensional. This type of consideration will form the basis for all that will be done here, once the basic definitions and the basic properties concerning partial groups, objective partial groups, and localities, have been laid down.

Finite localities were introduced by the first author in [Ch1], in order to give a positive solution to the question: Given a saturated fusion system \mathcal{F} on a finite p -group, does there exist a unique “classifying space” for \mathcal{F} ? The solution that was given in [Ch1] was closely tied to the specific goal, and did not allow for a complete development of ideas. The aim here is to attempt such a development, to thereby obtain the prerequisites for a version of finite group theory itself from a strictly “ p -local” point of view, and to extend that point of view beyond the finite context. Concomitantly, our aim is to enrich the theory of fusion systems and to broaden its scope, with the aim of enabling a deeper connection with the homotopy theory in which - through Bob Oliver’s proof [O1 and O2] of the Martino-Priddy conjecture - the entire enterprise has its roots.

A locality is something which need not be a group, but which can have plenty of subgroups. All of the groups to be considered here will be subgroups of localities, and a basic assumption throughout is that any subgroup of a locality should be countable and locally finite. This appears to be the natural degree of generality for a theory of “discrete” localities, but it seems obvious to us that further generalization is possible. At any rate, this basic assumption entails, for example, that the algebraic groups that will be encountered here are defined over subfields of algebraic closures of prime fields. It implies also that when we speak of a p -group P , we mean a countable, locally finite group all of whose elements have order a power of p . Such a p -group P need not have the property that is so important when working with finite p -groups: that the normalizer in P of a proper subgroup Q of P necessarily contains Q properly. It will become necessary to impose this “normalizer-increasing property” in Parts II and III, but in Part I we can get by without it. A weak version of this property (see 3.2 and 3.3 below) is already implied by the hypothesis of finite-dimensionality.

The division into Parts closely parallels the extent to which fusion systems on the one hand, and primes other than p on the other, are drawn into the developing picture. This Part I can be characterized by its having no direct involvement with fusion systems at all, and by there being no mention in it of p' -groups. The main results here, after the foundations for partial groups and localities have been laid down in the first three sections, concern partial normal subgroups of localities, and quotient localities.

Part II interweaves the development of the fusion system of a locality (\mathcal{L}, Δ, S) with the notion of “expansion” to a locality $(\mathcal{L}^+, \Delta^+, S)$ having the same fusion system and having a larger collection Δ^+ of objects. This material will provide the technical foundation for Part III, in which the focus is on the development of the generalized Fitting locality $F^*(\mathcal{L})$ of a locality \mathcal{L} . Readers who are familiar with the work of Michael Aschbacher on fusion systems (see especially [Asch]) will recognize that the path that we are taking is to a large extent guided by his work, but will also recognize the fundamental differences between his strictly fusion-theoretic setup and ours, and between the kinds of arguments that drive the development. The payoff will come in Part III with a category of what we shall call “regular” localities, among whose morphisms are embeddings $\lambda : \mathcal{H} \rightarrow \mathcal{L}$ such that $\text{Im}(\lambda)$ is normal in \mathcal{L} or such that $\text{Im}(\lambda)$ is the normalizer in \mathcal{L} of a p -subgroup of $F^*(\mathcal{L})$.

“Finite versions” of all three parts are currently available on the Math ArXiv [Ch2,3,4].

Let G be a group, and let $\mathbf{W}(G)$ be the free monoid on G . Thus, $\mathbf{W}(G)$ is the set of all words in the alphabet G , with the binary operation given by concatenation of words. The product $G \times G \rightarrow G$ extends, by generalized associativity, to a “product” $\Pi : \mathbf{W}(G) \rightarrow G$, whereby a word $w = (g_1, \dots, g_n) \in \mathbf{W}(G)$ is mapped to $g_1 \cdots g_n$. The inversion map on G induces an “inversion” on $\mathbf{W}(G)$, sending w to $(g_n^{-1}, \dots, g_1^{-1})$. In fact, one may easily replace the standard definition of “group” by a definition given in terms of Π and the inversion on $\mathbf{W}(G)$. One obtains the notion of *partial group* by restricting the domain of Π to a subset \mathbf{D} of $\mathbf{W}(G)$, where \mathbf{D} , the product, and the inversion, are required to satisfy conditions (see definition 1.1) that preserve the outlines of the strictly group-theoretic setup. When one looks at things in this way, a group is simply a partial group G having the property that $\mathbf{D} = \mathbf{W}(G)$.

The notions of partial subgroup and homomorphism of partial groups immediately suggest themselves, and a partial subgroup of a partial group \mathcal{L} may in fact be a group. We say that the partial group \mathcal{L} is “objective” (see definition 2.1) provided that the domain \mathbf{D} of the product is determined in a certain way by a collection Δ of subgroups of \mathcal{L} (the set of “objects”), and provided that Δ has a certain “closure” property. If also \mathcal{L} there exists $S \in \Delta$ such that Δ is a collection of subgroups of S then (\mathcal{L}, Δ, S) is a *pre-locality*. A locality is then defined to be a pre-locality (\mathcal{L}, Δ, S) such that S is a p -group which is maximal in the set (partially ordered by inclusion) of all p -subgroups of \mathcal{L} , and such that the groups $N_{\mathcal{L}}(P)$ for P in Δ are countable and locally finite.

The basic properties of partial groups, objective partial groups, and localities, are derived in sections 1 through 3. We then begin in section 4 to consider partial normal subgroups of localities in detail. One of the two key results in section 4 is the Frattini Lemma (4.11), which states that if $\mathcal{N} \trianglelefteq \mathcal{L}$ is a partial normal subgroup, then $\mathcal{L} = N_{\mathcal{L}}(S \cap \mathcal{N})\mathcal{N}$. The other is Stellmacher’s splitting lemma (4.12), which leads to the partition of \mathcal{L} into a collection of “maximal cosets” of \mathcal{N} , and to a partial group structure on the set \mathcal{L}/\mathcal{N} of maximal cosets. In section 5 it is shown that \mathcal{L}/\mathcal{N} is in fact a locality, and we obtain versions of the first Nöther isomorphism theorem and of its familiar consequences. In particular, the notions “partial normal subgroup” and “kernel

of a projection” are equivalent.

Section 6 concerns products of partial normal subgroups. The main results here (Theorems 6.7 and 6.8) owe a great deal to the treatment by Ellen Henke [He] in the case of finite localities.

Concerning notation: We adopt the group-theorist’s use of “ X^g ” to denote conjugation of a subset or element X of a group G by an element $g \in G$. Consistent with this practice, mappings will (almost always) be written to the right of their arguments.

The authors extends their appreciation to Bernd Stellmacher for suggesting and proving the splitting lemma (3.12), and for helpful suggestions regarding definition 1.1.

Section 1: Partial groups

The reader is asked to forget what a group is, and to trust that what was forgotten will soon be recovered.

For any set X write $\mathbf{W}(X)$ for the free monoid on X . Thus, an element of $\mathbf{W}(X)$ is a finite sequence of (or *word in*) the elements of X . The multiplication in $\mathbf{W}(X)$ consists of concatenation of words, to be denoted $u \circ v$. The *length* of the word (x_1, \dots, x_n) is n . The *empty word* is the word (\emptyset) of length 0. We make no distinction between X and the set of words of length 1.

Definition 1.1. Let \mathcal{L} be a non-empty set, let $\mathbf{W} = \mathbf{W}(\mathcal{L})$ be the free monoid on \mathcal{L} , and let \mathbf{D} be a subset of \mathbf{W} such that:

- (1) $\mathcal{L} \subseteq \mathbf{D}$ (i.e. \mathbf{D} contains all words of length 1), and

$$u \circ v \in \mathbf{D} \implies u, v \in \mathbf{D}.$$

Notice that since \mathcal{L} is non-empty, (1) implies that also the empty word is in \mathbf{D} .

A mapping $\Pi : \mathbf{D} \rightarrow \mathcal{L}$ is a *product* if:

- (2) Π restricts to the identity map on \mathcal{L} , and

- (3) $u \circ v \circ w \in \mathbf{D} \implies u \circ (\Pi(v)) \circ w \in \mathbf{D}$, and $\Pi(u \circ v \circ w) = \Pi(u \circ (\Pi(v)) \circ w)$.

An *inversion* on \mathcal{L} consists of an involutory bijection $x \mapsto x^{-1}$ on \mathcal{L} , together with the mapping $w \mapsto w^{-1}$ on \mathbf{W} given by

$$(x_1, \dots, x_n) \mapsto (x_n^{-1}, \dots, x_1^{-1}).$$

We say that \mathcal{L} , with the product $\Pi : \mathbf{D} \rightarrow \mathcal{L}$ and inversion $(-)^{-1}$, is a *partial group* if:

- (4) $w \in \mathbf{D} \implies w^{-1} \circ w \in \mathbf{D}$ and $\Pi(w^{-1} \circ w) = \mathbf{1}$,

where $\mathbf{1}$ denotes the image of the empty word under Π . (Notice that (1) and (4) yield $w^{-1} \in \mathbf{D}$ if $w \in \mathbf{D}$. As $(w^{-1})^{-1} = w$, condition (4) is then symmetric.)

Example 1.2. Let \mathcal{L} be the 3-element set $\{\mathbf{1}, a, b\}$ and let \mathbf{D} be the subset of $\mathbf{W}(\mathcal{L})$ consisting of all words w such that the word obtained from w by deleting all entries equal to $\mathbf{1}$ is an alternating string of a 's and b 's (of odd or even length, beginning with a or beginning with b). Define $\Pi : \mathbf{D} \rightarrow \mathcal{L}$ by the formula: $\Pi(w) = \mathbf{1}$ if the number of a -entries in w is equal to the number of b 's; $\Pi(w) = a$ if the number of a 's exceeds the number of b 's (necessarily by 1); and $\Pi(w) = b$ if the number of b 's exceeds the number of a 's. Define inversion on \mathcal{L} by $\mathbf{1}^{-1} = \mathbf{1}$, $a^{-1} = b$, and $b^{-1} = a$. It is then easy to check that \mathcal{L} , with these structures, is a partial group. In fact, \mathcal{L} is the “free partial group on one generator”, as will be made clear in 1.12.

It will be convenient to make the definition: a *group* is a partial group \mathcal{L} in which $\mathbf{W}(\mathcal{L}) = \mathbf{D}$. In order to distinguish between this definition and the usual one, we shall use the expression *binary group* for a set G with an associative binary operation, identity element, and inverses in the usual sense. The following lemma shows that the distinction is subtle.

Lemma 1.3.

- (a) *Let G be a binary group and let $\Pi : \mathbf{W}(G) \rightarrow G$ be the “multivariable product” on G given by $(g_1, \dots, g_n) \mapsto g_1 \cdots g_n$. Then G , together with Π and the inversion in G , is a partial group, with $\mathbf{D} = \mathbf{W}(G)$.*
- (b) *Let \mathcal{L} be a group; i.e. a partial group for which $\mathbf{W} = \mathbf{D}$. Then \mathcal{L} is a binary group with respect to the operation given by restricting Π to words of length 2, and with respect to the inversion in \mathcal{L} . Moreover, Π is then the multivariable product on \mathcal{L} defined as in (a).*

Proof. Point (a) is given by generalized associativity in the binary group G . Point (b) is a straightforward exercise, and is left to the reader. \square

We now list some elementary consequences of definition 1.1.

Lemma 1.4. *Let \mathcal{L} (with \mathbf{D} , Π , and the inversion) be a partial group.*

- (a) *Π is \mathbf{D} -multiplicative. That is, if $u \circ v$ is in \mathbf{D} then the word $(\Pi(u), \Pi(v))$ of length 2 is in \mathbf{D} , and*

$$\Pi(u \circ v) = \Pi(u)\Pi(v),$$

where “ $\Pi(u)\Pi(v)$ ” is an abbreviation for $\Pi((\Pi(u), \Pi(v)))$.

- (b) *Π is \mathbf{D} -associative. That is:*

$$u \circ v \circ w \in \mathbf{D} \implies \Pi(u \circ v)\Pi(w) = \Pi(u)\Pi(v \circ w).$$

- (c) *If $u \circ v \in \mathbf{D}$ then $u \circ (\mathbf{1}) \circ v \in \mathbf{D}$ and $\Pi(u \circ (\mathbf{1}) \circ v) = \Pi(u \circ v)$.*
- (d) *If $u \circ v \in \mathbf{D}$ then both $u^{-1} \circ u \circ v$ and $u \circ v \circ v^{-1}$ are in \mathbf{D} , $\Pi(u^{-1} \circ u \circ v) = \Pi(v)$, and $\Pi(u \circ v \circ v^{-1}) = \Pi(u)$.*
- (e) *The cancellation rule: If $u \circ v$, $u \circ w \in \mathbf{D}$, and $\Pi(u \circ v) = \Pi(u \circ w)$, then $\Pi(v) = \Pi(w)$ (and similarly for right cancellation).*

- (f) If $u \in \mathbf{D}$ then $u^{-1} \in \mathbf{D}$, and $\Pi(u^{-1}) = \Pi(u)^{-1}$. In particular, $\mathbf{1}^{-1} = \mathbf{1}$.
(g) The uncancellation rule: Suppose that both $u \circ v$ and $u \circ w$ are in \mathbf{D} and that $\Pi(v) = \Pi(w)$. Then $\Pi(u \circ v) = \Pi(u \circ w)$. (Similarly for right uncancellation.)

Proof. Let $u \circ v \in \mathbf{D}$. Then 1.1(3) applies to $(\emptyset) \circ u \circ v$ and yields $(\Pi(u)) \circ v \in \mathbf{D}$ with $\Pi(u \circ v) = \Pi((\Pi(u)) \circ v)$. Now apply 1.1(3) to $(\Pi(u)) \circ v \circ (\emptyset)$, to obtain (a).

Let $u \circ v \circ w \in \mathbf{D}$. Then $u \circ v$ and w are in \mathbf{D} by 1.1(1), and \mathbf{D} -multiplicativity yields $\Pi(u \circ v \circ w) = \Pi(u \circ v)\Pi(w)$. Similarly, $\Pi(u \circ v \circ w) = \Pi(u)\Pi(v \circ w)$, and (b) holds.

Since $\mathbf{1} = \Pi(\emptyset)$, point (c) is immediate from 1.1(3).

Let $u \circ v \in \mathbf{D}$. Then $v^{-1} \circ u^{-1} \circ u \circ v \in \mathbf{D}$ by 1.1(4), and then $u^{-1} \circ u \circ v \in \mathbf{D}$ by 1.1(1). Multiplicativity then yields

$$\Pi(u^{-1} \circ u \circ v) = \Pi(u^{-1} \circ u)\Pi(v) = \mathbf{1}\Pi(v) = \Pi(\emptyset)\Pi(v) = \Pi(\emptyset \circ v) = \Pi(v).$$

As $(w^{-1})^{-1} = w$ for any $w \in \mathbf{W}$, one obtains $w \circ w^{-1} \in \mathbf{D}$ for any $w \in \mathbf{D}$, and $\Pi(w \circ w^{-1}) = \mathbf{1}$. From this one easily completes the proof of (d).

Now let $u \circ v$ and $u \circ w$ be in \mathbf{D} , with $\Pi(u \circ v) = \Pi(u \circ w)$. Then (d) (together with multiplicativity and associativity, which will not be explicitly mentioned hereafter) yield $\Pi(v) = \Pi(u^{-1} \circ u \circ v) = \Pi(u^{-1})\Pi(u)\Pi(v) = \Pi(u^{-1})\Pi(u)\Pi(w) = \Pi(u^{-1} \circ u \circ w) = \Pi(w)$, and (e) holds.

Let $u \in \mathbf{D}$. Then $u \circ u^{-1} \in \mathbf{D}$, and then $\Pi(u)\Pi(u^{-1}) = \mathbf{1}$. But also $(\Pi(u), \Pi(u)^{-1}) \in \mathbf{D}$, and $\Pi(u)\Pi(u)^{-1} = \mathbf{1}$. Now (f) follows by 1.1(2) and cancellation.

Let u, v, w be as in (g). Then $u^{-1} \circ u \circ v$ and $u^{-1} \circ u \circ w$ are in \mathbf{D} by (d). By two applications of (d), $\Pi(u^{-1} \circ u \circ v) = \Pi(v) = \Pi(w) = \Pi(u^{-1} \circ u \circ w)$, so $\Pi(u \circ v) = \Pi(u \circ w)$ by (e), and (g) holds. \square

It will often be convenient to eliminate the symbol “ Π ” and to speak of “the product $g_1 \cdots g_n$ ” instead of $\Pi(g_1, \dots, g_n)$. More generally, if $\{X_i\}_{1 \leq i \leq n}$ is a collection of subsets of \mathcal{L} then the “product set $X_1 \cdots X_n$ ” is by definition the image under Π of the set of words $(g_1, \dots, g_n) \in \mathbf{D}$ such that $g_i \in X_i$ for all i . If $X_i = \{g_i\}$ is a singleton then we may write g_i in place of X_i in such a product. Thus, for example, the product $g^{-1}Xg$ stands for the set of all $\Pi(g^{-1}, x, g)$ with $(g^{-1}, x, g) \in \mathbf{D}$, and with $x \in X$.

A Word of Urgent Warning: In writing products in the above way one may be drawn into imagining that associativity holds in a stronger sense than that which is given by 1.4(b). This is an error that is to be avoided. For example one should not suppose, if $(f, g, h) \in \mathbf{W}$, and both (f, g) and (fg, h) are in \mathbf{D} , that (f, g, h) is in \mathbf{D} . That is, it may be that “the product fgh ” is undefined, even though the product $(fg)h$ is defined. Of course, one is tempted to simply extend the domain \mathbf{D} to include such triples (f, g, h) , and to “define” the product fgh to be $(fg)h$. The trouble is that it may also be the case that gh and $f(gh)$ are defined, but that $(fg)h \neq f(gh)$.

For \mathcal{L} a partial group and $g \in \mathcal{L}$, write $\mathbf{D}(g)$ for the set of all $x \in \mathcal{L}$ such that the product $g^{-1}xg$ is defined. There is then a mapping

$$c_g : \mathbf{D}(g) \rightarrow \mathcal{L}$$

given by $x \mapsto g^{-1}xg$ (and called *conjugation by g*). Our preference is for right-hand notation for mappings, so we write

$$x \mapsto (x)c_g \quad \text{or} \quad x \mapsto x^g$$

for conjugation by g .

The following result provides an illustration of the preceding notational conventions, and introduces a theme which will be developed further as we pass from partial groups to objective partial groups, localities, and (in Part III) regular localities.

Lemma 1.5. *Let \mathcal{L} be a partial group, and let $f, g \in \mathcal{L}$.*

- (a) *Suppose that the products fg and gf are defined and that $fg = gf$. Suppose further that $f \in \mathbf{D}(g)$. Then $f^g = f$.*
- (b) *Suppose that $f \in \mathbf{D}(g)$, $g \in \mathbf{D}(f)$, and $f^g = f$. Then $fg = gf$ and $g^f = g$.*

Proof. (a): We are given $(f, g) \in \mathbf{D}$, so $(f^{-1}, f, g) \in \mathbf{D}$ and $\Pi(f^{-1}, f, g) = g$, by 1.4(d) and \mathbf{D} -associativity. We are given also $f \in \mathbf{D}(g)$ and $fg = gf$, so

$$f^g = \Pi(g^{-1}, f, g) = \Pi((g^{-1}, fg) = \Pi(g^{-1}, gf) = \Pi(g^{-1}, g, f) = f.$$

(b): As $(g^{-1}, f, g) \in \mathbf{D}$ we obtain $(f, g) \in \mathbf{D}$ from 1.1(1). As $(f^{-1}, g, f) \in \mathbf{D}$, we get $(g, g^{-1}, f, g) \in \mathbf{D}$ by 1.4(d). Then \mathbf{D} -associativity yields $fg = \Pi(g, g^{-1}, f, g) = gf^g$. As $f^g = f$ by hypothesis, we obtain $fg = gf$. Finally, since (f^{-1}, f, g) and (f^{-1}, g, f) are in \mathbf{D} the uncancellation rule yields $f^{-1}fg = f^{-1}gf$, and so $g^f = g$. \square

Notation. From now on, in any given partial group \mathcal{L} , usage of the symbol “ x^g ” shall be taken to imply $x \in \mathbf{D}(g)$. More generally, for X a subset of \mathcal{L} and $g \in \mathcal{L}$, usage of “ X^g ” shall be taken to mean that $X \subseteq \mathbf{D}(g)$; whereupon X^g is by definition the set of all x^g with $x \in X$.

At this early point, and in the context of arbitrary partial groups, one can say very little about the maps c_g . The cancellation rule 1.4(e) implies that each c_g is injective, but beyond that, the following lemma may be the best that can be obtained.

Lemma 1.6. *Let \mathcal{L} be a partial group and let $g \in \mathcal{L}$. Then the following hold.*

- (a) $\mathbf{1} \in \mathbf{D}(g)$ and $\mathbf{1}^g = \mathbf{1}$.
- (b) $\mathbf{D}(g)$ is closed under inversion, and $(x^{-1})^g = (x^g)^{-1}$ for all $x \in \mathbf{D}(g)$.
- (c) c_g is a bijection $\mathbf{D}(g) \rightarrow \mathbf{D}(g^{-1})$, and $c_{g^{-1}} = (c_g)^{-1}$.
- (d) $\mathcal{L} = \mathbf{D}(\mathbf{1})$, and $x^{\mathbf{1}} = x$ for each $x \in \mathcal{L}$.

Proof. By 1.1(4), $g \circ \emptyset \circ g^{-1} = g \circ g^{-1} \in \mathbf{D}$, so $\mathbf{1} \in \mathbf{D}(g)$ and then $\mathbf{1}^g = \mathbf{1}$ by 1.4(c). Thus (a) holds. Now let $x \in \mathbf{D}(g)$ and set $w = (g^{-1}, x, g)$. Then $w \in \mathbf{D}$, and $w^{-1} = (g^{-1}, x^{-1}, g)$ by definition in 1.1. Then 1.1(4) yields $w^{-1} \circ w \in \mathbf{D}$, and so $w^{-1} \in \mathbf{D}$ by 1.1(1). This shows that $\mathbf{D}(g)$ is closed under inversion. Also, 1.1(4) yields $\mathbf{1} =$

$\Pi(w^{-1} \circ w) = (x^{-1})^g x^g$, and then $(x^{-1})^g = (x^g)^{-1}$ by 1.4(f). This completes the proof of (b).

As $w \in \mathbf{D}$, 1.4(d) implies that $g \circ w$ and then $g \circ w \circ g^{-1}$ are in \mathbf{D} . Now 1.1(3) and two applications of 1.4(d) yield

$$gx^g g^{-1} = \Pi(g, g^{-1}, x, g, g^{-1}) = \Pi((g, g^{-1}, x) \circ g \circ g^{-1}) = \Pi(g, g^{-1}, x) = x.$$

Thus $x^g \in \mathbf{D}(g^{-1})$ with $(x^g)^{g^{-1}} = x$, and thus (c) holds.

Finally, $\mathbf{1} = \mathbf{1}^{-1}$ by 1.4(f), and $\emptyset \circ x \circ \emptyset = x \in \mathbf{D}$ for any $x \in \mathcal{L}$, proving (d). \square

Definition 1.7. Let \mathcal{L} be a partial group and let \mathcal{H} be a non-empty subset of \mathcal{L} . Then \mathcal{H} is a *partial subgroup* of \mathcal{L} (denoted $\mathcal{H} \leq \mathcal{L}$) if \mathcal{H} is closed under inversion ($g \in \mathcal{H}$ implies $g^{-1} \in \mathcal{H}$) and with respect to products. The latter condition means that $\Pi(w) \in \mathcal{H}$ whenever $w \in \mathbf{W}(\mathcal{H}) \cap \mathbf{D}$. A partial subgroup \mathcal{N} of \mathcal{L} is *normal* in \mathcal{L} (denoted $\mathcal{N} \trianglelefteq \mathcal{L}$) if $x^g \in \mathcal{N}$ for all $x \in \mathcal{N}$ and all $g \in \mathcal{L}$ for which $x \in \mathbf{D}(g)$. We say that \mathcal{H} is a *subgroup* of \mathcal{L} if $\mathcal{H} \leq \mathcal{L}$ and $\mathbf{W}(\mathcal{H}) \subseteq \mathbf{D}$.

An equivalent way to state the condition for normality is to say that the partial subgroup \mathcal{N} of \mathcal{L} is normal in \mathcal{L} if $g^{-1}\mathcal{N}g \subseteq \mathcal{N}$ for all $g \in \mathcal{L}$. (This formulation relies on a notational convention introduced above for interpreting product sets XYZ .)

We leave it to the reader to check that if $\mathcal{H} \leq \mathcal{L}$ then \mathcal{H} is a partial group, with $\mathbf{D}(\mathcal{H}) = \mathbf{W}(\mathcal{H}) \cap \mathbf{D}(\mathcal{L})$.

Lemma 1.8. Let \mathcal{H} and \mathcal{K} be partial subgroups of a partial group \mathcal{L} , and let $\{\mathcal{H}_i\}_{i \in I}$ be a set of partial subgroups of \mathcal{L} .

- (a) Each partial subgroup of \mathcal{H} is a partial subgroup of \mathcal{L} .
- (b) Each partial subgroup of \mathcal{L} which is contained in \mathcal{H} is a partial subgroup of \mathcal{H} .
- (c) If \mathcal{H} is a subgroup of \mathcal{L} then $\mathcal{H} \cap \mathcal{K}$ is a subgroup of both \mathcal{H} and \mathcal{K} .
- (d) Suppose $\mathcal{K} \trianglelefteq \mathcal{L}$. Then $\mathcal{H} \cap \mathcal{K} \trianglelefteq \mathcal{H}$. Moreover, $\mathcal{H} \cap \mathcal{K}$ is a normal subgroup of \mathcal{H} if \mathcal{H} is a subgroup of \mathcal{L} .
- (e) $\bigcap \{\mathcal{H}_i \mid i \in I\}$ is a partial subgroup of \mathcal{L} , and is a partial normal subgroup of \mathcal{L} if $\mathcal{H}_i \trianglelefteq \mathcal{L}$ for all i .

Proof. One observes that in all of the points (a) through (e) the requisite closure with respect to inversion obtains. Thus, we need only be concerned with products.

(a) Let $\mathcal{E} \leq \mathcal{H}$ be a partial subgroup of \mathcal{H} . Then

$$\mathbf{D}(\mathcal{E}) = \mathbf{W}(\mathcal{E}) \cap \mathbf{D}(\mathcal{H}) = \mathbf{W}(\mathcal{E}) \cap (\mathbf{W}(\mathcal{H}) \cap \mathbf{D}(\mathcal{L})) = \mathbf{W}(\mathcal{E}) \cap \mathbf{D}(\mathcal{L}),$$

and (a) follows.

(b) Suppose $\mathcal{K} \subseteq \mathcal{H}$ and let $w \in \mathbf{W}(\mathcal{K}) \cap \mathbf{D}(\mathcal{H})$. As $\mathbf{D}(\mathcal{H}) \leq \mathbf{D}(\mathcal{L})$, and since $\mathcal{K} \leq \mathcal{L}$ by hypothesis, we obtain $\Pi(w) \in \mathcal{K}$.

(c) Assuming now that \mathcal{H} is a subgroup of \mathcal{L} , we have $\mathbf{W}(\mathcal{H}) \subseteq \mathbf{D}(\mathcal{L})$, and then $\mathbf{D}(\mathcal{H} \cap \mathcal{K}) \subseteq \mathbf{D}(\mathcal{H}) \cap \mathbf{D}(\mathcal{K})$, so that $\mathcal{H} \cap \mathcal{K}$ is a subgroup of both \mathcal{H} and \mathcal{K} .

(d) Let $\mathcal{K} \trianglelefteq \mathcal{L}$ and let $x \in \mathcal{H} \cap \mathcal{K}$ and $h \in \mathcal{H}$ with $(h^{-1}, x, h) \in \mathbf{D}(\mathcal{H})$. Then $(h^{-1}, x, h) \in \mathbf{D}(\mathcal{L})$, and $x^h \in \mathcal{K}$. As $\mathcal{H} \leq \mathcal{L}$ we have also $x^h \in \mathcal{H}$, and so $\mathcal{H} \cap \mathcal{K} \trianglelefteq \mathcal{H}$. Now suppose further that \mathcal{H} is a subgroup of \mathcal{L} . That is, assume that $\mathbf{W}(\mathcal{H}) \subseteq \mathbf{D}(\mathcal{L})$. Then $\mathbf{W}(\mathcal{H} \cap \mathcal{K}) \subseteq \mathbf{D}(\mathcal{L})$, hence $\mathcal{H} \cap \mathcal{K}$ is a subgroup of \mathcal{H} , and evidently a normal subgroup.

(e) Set $\mathcal{X} = \bigcap \{\mathcal{H}_i\}_{i \in I}$. Then $\Pi(w) \in \mathcal{X}$ for all $w \in \mathbf{W}(\mathcal{X}) \cap \mathbf{D}(\mathcal{L})$, and so $\mathcal{X} \leq \mathcal{L}$. The last part of (e) may be left to the reader. \square

For any subset X of a partial group \mathcal{L} define the partial subgroup $\langle X \rangle$ of \mathcal{L} *generated by* X to be the intersection of the set of all partial subgroups of \mathcal{L} containing X . Then $\langle X \rangle$ is itself a partial subgroup of \mathcal{L} by 1.8(e).

Lemma 1.9. *Let X be a subset of \mathcal{L} such that X is closed under inversion. Set $X_0 = X$ and recursively define X_n for $n > 0$ by*

$$X_n = \{\Pi(w) \mid w \in \mathbf{W}(X_{n-1}) \cap \mathbf{D}\}.$$

Then $\langle X \rangle = \bigcup \{X_n\}_{n \geq 0}$.

Proof. Let Y be the union of the sets X_i . Each X_i is closed under inversion by 1.4(f), and $Y \neq \emptyset$ since $\mathbf{1} = \Pi(\emptyset)$. Since Y is closed under products, by construction, we get $Y \leq \langle X \rangle$, and then $Y = \langle X \rangle$ by the definition of $\langle X \rangle$. \square

Lemma 1.10 (Dedekind Lemma). *Let \mathcal{H} , \mathcal{K} , and \mathcal{A} be partial subgroups of a partial group \mathcal{L} , and assume that $\mathcal{H}\mathcal{K}$ is a partial subgroup of \mathcal{L} .*

- (a) *If $\mathcal{K} \leq \mathcal{A}$ then $\mathcal{A} = (\mathcal{A} \cap \mathcal{H})\mathcal{K}$.*
- (b) *If $\mathcal{H} \leq \mathcal{A}$ then $\mathcal{A} = \mathcal{H}(\mathcal{A} \cap \mathcal{K})$.*

Proof. The proof is identical to the proof for binary groups, and is left to the reader. \square

Definition 1.11. Let \mathcal{L} and \mathcal{L}' be partial groups, let $\beta : \mathcal{L} \rightarrow \mathcal{L}'$ be a mapping, and let $\beta^* : \mathbf{W} \rightarrow \mathbf{W}'$ be the induced mapping of free monoids. Then β is a *homomorphism (of partial groups)* if:

- (H1) $\mathbf{D}\beta^* \subseteq \mathbf{D}'$, and
- (H2) $(\Pi(w))\beta = \Pi'(w\beta^*)$ for all $w \in \mathbf{D}$.

The *kernel* of β is the set $\text{Ker}(\beta)$ of all $g \in \mathcal{L}$ such that $g\beta = \mathbf{1}'$. We say that β is an *isomorphism* if there exists a homomorphism $\beta' : \mathcal{L}' \rightarrow \mathcal{L}$ such that $\beta \circ \beta'$ and $\beta' \circ \beta$ are identity mappings. (Equivalently, β is an isomorphism if β is bijective and $\mathbf{D}\beta = \mathbf{D}'$.)

Example 1.12. Let $\mathcal{L} = \{\mathbf{1}, a, b\}$ be the partial group from example 1.2, let \mathcal{L}' be any partial group, and let $x \in \mathcal{L}'$. Then the mapping $\beta : \mathcal{L} \rightarrow \mathcal{L}'$ given by

$$\mathbf{1} \mapsto \mathbf{1}', \quad a \mapsto x, \quad b \mapsto x^{-1}.$$

is a homomorphism. In fact, β is the unique homomorphism $\mathcal{L} \rightarrow \mathcal{L}'$ which maps a to x , by the following lemma. Thus, \mathcal{L} is the (unique up to a unique invertible homomorphism) *free partial group* on one generator. Free partial groups in general can be obtained as “free products” of copies of \mathcal{L} (see 1.17 below).

Lemma 1.13. *Let $\beta : \mathcal{L} \rightarrow \mathcal{L}'$ be a homomorphism of partial groups. Then $\mathbf{1}\beta = \mathbf{1}'$, and $(g^{-1})\beta = (g\beta)^{-1}$ for all $g \in \mathcal{L}$.*

Proof. Since $\mathbf{1}\mathbf{1} = \mathbf{1}$, (H1) and (H2) yield $\mathbf{1}\beta = (\mathbf{1}\mathbf{1})\beta = (\mathbf{1}\beta)(\mathbf{1}\beta)$, and then $\mathbf{1}\beta = \mathbf{1}'$ by left or right cancellation. Since $(g, g^{-1}) \in \mathbf{D}$ for any $g \in \mathcal{L}$ by 1.4(d), (H1) yields $(g\beta, (g^{-1})\beta) \in \mathbf{D}'$, and then $\mathbf{1}\beta = (gg^{-1})\beta = (g\beta)((g^{-1})\beta)$ by (H2). As $\mathbf{1}\beta = \mathbf{1}' = (g\beta)(g\beta)^{-1}$, left cancellation yields $(g^{-1})\beta = (g\beta)^{-1}$. \square

Lemma 1.14. *Let $\beta : \mathcal{L} \rightarrow \mathcal{L}'$ be a homomorphism of partial groups, and set $\mathcal{N} = \text{Ker}(\beta)$. Then \mathcal{N} is a partial normal subgroup of \mathcal{L} .*

Proof. By 1.13 \mathcal{N} is closed under inversion. For w in $\mathbf{W}(\mathcal{N}) \cap \mathbf{D}$ the map $\beta^* : \mathbf{W} \rightarrow \mathbf{W}'$ sends w to a word of the form $(\mathbf{1}', \dots, \mathbf{1}')$. Then $\Pi'(w\beta^*) = \mathbf{1}'$, and thus $\Pi(w) \in \mathcal{N}$ and \mathcal{N} is a partial subgroup of \mathcal{L} . Now let $f \in \mathcal{L}$ and let $g \in \mathcal{N} \cap \mathbf{D}(f)$. Then

$$(f^{-1}, g, f)\beta^* = ((f\beta)^{-1}, \mathbf{1}', f\beta) \quad (\text{by 1.13}),$$

so that

$$(g^f)\beta = \Pi'((f^{-1}, g, f)\beta^*) = \Pi'(f\beta)^{-1}, \mathbf{1}', f\beta = \mathbf{1}'.$$

Thus $\mathcal{N} \trianglelefteq \mathcal{L}$. \square

It will be shown later (cf. 4.6) that partial normal subgroups of “localities” are always kernels of homomorphisms.

Lemma 1.15. *Let $\beta : \mathcal{L} \rightarrow \mathcal{L}'$ be a homomorphism of partial groups and let M be a subgroup of \mathcal{L} . Then $M\beta$ is a subgroup of \mathcal{L}' . \square*

Proof. We are given $\mathbf{W}(M) \subseteq \mathbf{D}(\mathcal{L})$, so β^* maps $\mathbf{W}(M)$ into $\mathbf{D}(\mathcal{L}')$. (Note, however, example 1.12.) \square

Lemma 1.16. *Let G and G' be groups (and hence also binary groups in the sense of 1.3). A map $\alpha : G \rightarrow G'$ is a homomorphism of partial groups if and only if α is a homomorphism of binary groups.*

Proof. We leave to the reader the proof that if α is a homomorphism of partial groups then α is a homomorphism of binary groups. Now suppose that α is a homomorphism of binary groups. As $\mathbf{W}(G) = \mathbf{D}(G)$ (and similarly for G' , it is immediate that α^* maps $\mathbf{D}(G)$ into $\mathbf{D}(G')$. Assume that α is not a homomorphism of partial groups and let $w \in \mathbf{D}(G)$ be of minimal length subject to $\Pi'(w\alpha^*) \neq (\Pi(w))\alpha$. Then $n > 1$ and we can write $w = u \circ v$ with both u and v non-empty. Then

$$\Pi'(w\alpha^*) = \Pi'(u\alpha^* \circ v\alpha^*) = \Pi'(u\alpha^*)\Pi'(v\alpha^*) = ((\Pi(u))\alpha)((\Pi(v))\alpha) = (\Pi(u)\Pi(v))\alpha,$$

as α is a homomorphism of binary groups. Since $(\Pi(u)\Pi(v))\alpha = (\Pi(w))\alpha$, the proof is complete. \square

By a *pointed set* we mean a pointed topological space having the discrete topology. The pointed sets form a full subcategory of the category of topological spaces, and this subcategory is easily seen to possess all limits and colimits. We regard a partial group as a pointed set in which the base-point is the identity element.

Theorem 1.17. *Let $Part$ be the category of partial groups (and homomorphisms of partial groups), and let Set_* be the category of pointed sets. Then $Part$ has all limits and colimits, and these are preserved by the forgetful functor $Part \rightarrow Set_*$.*

Proof. We outline a proof only for colimits, leaving the details concerning limits to the reader.

Let \mathcal{C} be a small category, and let $F : \mathcal{C} \rightarrow Part$ be a covariant functor. Write \mathcal{L}_a for $F(a)$, a an object of \mathcal{C} , and write $\lambda_x : \mathcal{L}_b \rightarrow \mathcal{L}_a$ for $F(x)$, where $x : a \rightarrow b$ is a morphism of \mathcal{C} . Let \mathcal{L} be the set-theoretic colimit of F . Thus, \mathcal{L} is the disjoint union of the partial groups \mathcal{L}_a taken over all $a \in Ob(\mathcal{C})$, modulo the equivalence relation \sim generated by the requirement:

(*) For each \mathcal{C} -morphism $x : a \rightarrow b$ and each $g \in \mathcal{L}_b$, we have $g \sim g\lambda_x$.

For each object a of \mathcal{C} write \mathbf{D}_a for $\mathbf{D}(\mathcal{L}_a)$. Let \mathbf{D} be the set of all $w \in \mathbf{W}(\mathcal{L})$ for which there exists $a \in Ob(\mathcal{C})$ and $\tilde{w} = (g_1, \dots, g_n) \in \mathbf{D}_a$ such that w is the image of \tilde{w} under the component-wise application of \sim . Then \mathbf{D} contains all words of length 1 in $\mathbf{W}(\mathcal{L})$, and one observes that if $w \in \mathbf{D}$ then any segment of w is in \mathbf{D} . That is, \mathbf{D} satisfies the condition 1.1(1) in the definition of partial group.

There is a diagram

$$\bigsqcup \{\mathbf{D}(\mathcal{L}_a)\} \xrightarrow{\tilde{\Pi}} \bigsqcup \{\mathcal{L}_a\} \xrightarrow{\rho} \mathcal{L}$$

where $\tilde{\Pi}$ is the disjoint union of the products $\Pi_a : \mathbf{D}_a \rightarrow \mathcal{L}_a$, and ρ is the quotient map given by \sim . Let

$$\bigsqcup \{\mathbf{D}(\mathcal{L}_a)\} \xrightarrow{\rho^*} \mathbf{D}$$

be the component-wise quotient map. For $x \in Mor_{\mathcal{C}}(a, b)$, the condition that λ_x is a homomorphism of partial groups implies that $\tilde{\Pi} \circ \rho$ factors through ρ^* to a mapping $\Pi : \mathbf{D} \rightarrow \mathcal{L}$. That Π satisfies 1.1(2) and 1.1(3) is then immediate.

The inversion maps on the partial groups \mathcal{L}_a yield an involutory bijection on $\bigsqcup \{\mathcal{L}_a\}$, and 1.13 implies that this bijection factors through ρ to an involutory bijection $g \mapsto g^{-1}$ on \mathcal{L} . The reader may now verify the condition 1.1(4) for \mathcal{L} , in order to complete the proof that \mathcal{L} , with Π and the inversion, is a partial group. One also verifies that the maps $\rho_a : \mathcal{L}_a \rightarrow \mathcal{L}$ given by restricting ρ are homomorphisms of partial groups. Since \mathcal{L} is a colimit as a pointed set, we conclude that \mathcal{L} is a colimit in the category of partial groups.

The situation for limits (i.e. where $F : \mathcal{C} \rightarrow Part$ is contravariant) is entirely analogous to the fore-going. That is, one forms the set-theoretic limit \mathcal{L} (the set of all tuples $(g_a)_{a \in Ob(\mathcal{C})}$ such that $g_b = g_a \lambda_x$ for each \mathcal{C} -morphism $x : a \rightarrow b$), and one endows \mathcal{L} with a product and an inversion map. The domain \mathbf{D} is in this case the set of all words $w \in \mathbf{W}(\mathcal{L})$ such that, for each of the projection maps $\mathcal{L} \xrightarrow{\pi_a} \mathcal{L}_a$, we have $w\pi_a^* \in \mathbf{D}_a$. One then defines $\Pi : \mathbf{D} \rightarrow \mathcal{L}$ by

$$((g_{a,1}, \dots, (g_{a,n}) \mapsto (\Pi_a(g_{a,1}, \dots, g_{a,n}).$$

The inversion on \mathcal{L} is of course given by $(g_a) \mapsto (g_a^{-1})$, and then the verification of the conditions in definition 1.1 is straightforward. \square

Section 2: Objective partial groups and pre-localities

Recall the convention: if X is a subset of the partial group \mathcal{L} , and $g \in \mathcal{L}$, then any statement involving the expression “ X^g ” is to be understood as carrying the assumption that $X \subseteq \mathbf{D}(g)$. Thus, the statement “ $X^g = Y$ ” means: $(g^{-1}, x, g) \in \mathbf{D}$ for all $x \in X$, and Y is the set of products $g^{-1}xg$ with $x \in X$.

Definition 2.1. Let \mathcal{L} be a partial group and let Δ be a collection of subgroups of \mathcal{L} . Define \mathbf{D}_Δ to be the set of all $w = (g_1, \dots, g_n) \in \mathbf{W}(\mathcal{L})$ such that:

- (*) there exists $(X_0, \dots, X_n) \in \mathbf{W}(\Delta)$ with $(X_{i-1})^{g_i} = X_i$ for all i ($1 \leq i \leq n$).

Then (\mathcal{L}, Δ) is an *objective partial group* (in which Δ is the set of *objects*), if the following two conditions hold.

- (O1) $\mathbf{D} = \mathbf{D}_\Delta$.
- (O2) Whenever X and Y are in Δ and $g \in \mathcal{L}$ such that X^g is a subgroup of Y , then every subgroup of Y containing X^g is in Δ .

We emphasize that the condition (O2) requires more than that X^g be a *subset* of Y , in order to conclude that overgroups of X in Y are objects. This is a non-vacuous distinction, since the conjugation map $c_g : X \rightarrow X^g$ need not send X to a subgroup (or even a partial subgroup) of \mathcal{L} , in a general partial group.

Example 2.2. Let G be a group, let S be a subgroup of G , and let Δ be a collection of subgroups of S such that $S \in \Delta$. Assume that Δ satisfies (O2). That is, assume that $Y \in \Delta$ for every subgroup Y of S such that $X^g \leq Y$ for some $X \in \Delta$ and some $g \in G$. Let \mathcal{L} be the set of all $g \in G$ such that $S \cap S^g \in \Delta$, and let \mathbf{D} be the subset \mathbf{D}_Δ of $\mathbf{W}(\mathcal{L})$. Then \mathcal{L} is a partial group (via the multivariable product in G and the inversion in G), and (\mathcal{L}, Δ) is an objective partial group. Specifically:

- (a) If $\Delta = \{S\}$ then $\mathcal{L} = N_G(S)$ (and so \mathcal{L} is a group in this case).
- (b) Take $G = O_4^+(2)$. Thus, G is a semidirect product $V \rtimes S$ where V is elementary abelian of order 9 and S is a dihedral group of order 8 acting faithfully on V . Let Δ be the set of all non-identity subgroups of S . One may check that $S \cap S^g \in \Delta$ for all $g \in G$, and hence $\mathcal{L} = G$ (as *sets*). But \mathcal{L} is not a group, as $\mathbf{D}_\Delta \neq \mathbf{W}(G)$.
- (c) Take $G = GL_3(2)$, $S \in Syl_2(G)$, and let M_1 and M_2 be the two maximal subgroups of G containing S . Set $P_i = O_2(M_i)$ and set $\Delta = \{S, P_1, P_2\}$. Then $\mathcal{L} = M_1 \cup M_2$ (in fact the “free amalgamated product” of M_1 and M_2 over S in the category of partial groups). On the other hand, if Δ is taken to be the set of all non-identity subgroups of S then \mathcal{L} is somewhat more complicated. Its underlying set is $M_1 M_2 \cup M_2 M_1$.

In an objective partial group (\mathcal{L}, Δ) we say that the word $w = (g_1, \dots, g_n)$ is *in \mathbf{D} via (X_0, \dots, X_n)* if the condition (*) in 2.1 applies specifically to w and (X_0, \dots, X_n) .

We may also say, more simply, that w is in \mathbf{D} via X_0 , since the sequence (X_0, \dots, X_n) is determined by w and X_0 .

For any partial group \mathcal{L} and subgroups X, Y of \mathcal{L} , set

$$N_{\mathcal{L}}(X, Y) = \{g \in \mathcal{L} \mid X \subseteq \mathbf{D}(g), X^g \subseteq Y\},$$

and set

$$N_{\mathcal{L}}(X) = \{g \in \mathcal{L} \mid X^g = X\}.$$

Lemma 2.3. *Let (\mathcal{L}, Δ) be an objective partial group.*

- (a) $N_{\mathcal{L}}(X)$ is a subgroup of \mathcal{L} for each $X \in \Delta$.
- (b) Let $g \in \mathcal{L}$ and let $X \in \Delta$ with $Y := X^g \in \Delta$. Then $N_{\mathcal{L}}(X) \subseteq \mathbf{D}(g)$, and

$$c_g : N_{\mathcal{L}}(X) \rightarrow N_{\mathcal{L}}(Y)$$

is an isomorphism of groups. More generally:

- (c) Let $w = (g_1, \dots, g_n) \in \mathbf{D}$ via (X_0, \dots, X_n) . Then

$$c_{g_1} \circ \dots \circ c_{g_n} = c_{\Pi(w)}$$

as isomorphisms from $N_G(X_0)$ to $N_G(X_n)$.

Proof. (a) Let $X \in \Delta$ and let $u \in \mathbf{W}(N_{\mathcal{L}}(X))$. Then $u \in \mathbf{D}$ via X , $\mathbf{1} \in N_{\mathcal{L}}(X)$ (1.6(d)), and $N_{\mathcal{L}}(X)^{-1} = N_{\mathcal{L}}(X)$ (1.6(c)).

(b) Let $x, y \in N_{\mathcal{L}}(X)$ and set $v = (g^{-1}, x, g, g^{-1}, y, g)$. Then $v \in \mathbf{D}$ via Y , and then $\Pi(v) = (xy)^g = x^g y^g$ (using points (a) and (b) of 1.4). Thus, the conjugation map $c_g : N_{\mathcal{L}}(X) \rightarrow N_{\mathcal{L}}(Y)$ is a homomorphism of binary groups (see 1.3), and hence a homomorphism of partial groups (1.16). Since $c_{g^{-1}} = c_g^{-1}$ by 1.6(c), c_g is an isomorphism of groups.

(c) Let $x \in N_{\mathcal{L}}(X_0)$, set $u_x = w^{-1} \circ (x) \circ w$, and observe that $u_x \in \mathbf{D}$ via X_n . Then $\Pi(u_x)$ can be written as $(\dots (x)^{g_1} \dots)^{g_n}$, and this yields (c). \square

The next lemma provides two basic computational tools.

Lemma 2.4. *Let (\mathcal{L}, Δ) be an objective partial group.*

- (a) Let $(a, b, c) \in \mathbf{D}$ and set $d = abc$. Then $bc = a^{-1}d$ and $ab = dc^{-1}$ (and all of these products are defined).
- (b) Let $(f, g) \in \mathbf{D}$ and let $X \in \Delta$. Suppose that both X^f and X^{fg} are in Δ . Then $X^{fg} = (X^f)^g$.

Proof. Point (a) is a fact concerning partial groups in general, and is immediate from 1.4(c). Now consider the setup in (b). As $(f, g) \in \Delta$ we have also $(f^{-1}, f, g) \in \Delta$, and $g = \Pi(f^{-1}, f, g) = f^{-1}(fg)$. Now observe that $(f^{-1}, fg) \in \mathbf{D}$ via P^f , and apply 2.3(c) to obtain $P^{fg} = ((P^f)^{f^{-1}})^{fg} = (P^f)^g$. \square

The following result is a version of lemma 1.5(b) for objective partial groups. The hypothesis is weaker than that of 1.5(b), and the conclusion is stronger.

Lemma 2.5. *Let (\mathcal{L}, Δ) be an objective partial group and let $f, g \in \mathcal{L}$. Suppose that $f^g = f$. Then $fg = gf$ and $g^f = g$.*

Proof. Suppose that $(g^{-1}, f, g) \in \mathbf{D}$ via (P_0, P_1, P_2, P_3) . One then has the following commutative square of conjugation maps, in which the arrows are labeled by elements that perform the conjugation.

$$\begin{array}{ccc} P_2 & \xrightarrow{g} & P_3 \\ f \uparrow & & \uparrow f^g \\ P_1 & \xrightarrow{g} & P_0 \end{array}$$

Now assume that $f^g = f$. Since any of the arrows in the diagram may be reversed using 1.6(c), one reads off that $(f^{-1}, g, f) \in \mathbf{D}$ via P_2 . Then $g^f = g$ and $fg = gf$ by 1.5(b). \square

Definition 2.6. A *pre-locality* is an objective partial group (\mathcal{L}, Δ) such that Δ is a collection of subgroups of some $S \in \Delta$.

In order to emphasize the role of S , we shall write (\mathcal{L}, Δ, S) to indicate a pre-locality whose underlying objective partial group is (\mathcal{L}, Δ) and whose unique maximal object is S .

Lemma 2.7. *Let (\mathcal{L}, Δ, S) be a pre-locality. Then \mathbf{D} is an $N_{\mathcal{L}}(S)$ -biset. That is, if $w \in \mathbf{D}$ and $g, h \in N_{\mathcal{L}}(S)$, then $(g) \circ w \circ (h) \in \mathbf{D}$. In particular, $N_{\mathcal{L}}(S)$ acts on \mathcal{L} by conjugation.*

Proof. If $w \in \mathbf{D}$ via $X \in \Delta$, then $(g) \circ w \circ (h) \in \mathbf{D}$ via $X^{g^{-1}}$. The action of $N_{\mathcal{L}}(S)$ on \mathcal{L} is then given by restricting to the case where $g = h^{-1}$ and where w is of length 1. \square

The following result and its corollary are fundamental to the entire enterprise. The proof is due to Bernd Stellmacher.

Proposition 2.8. *Let (\mathcal{L}, Δ, S) be a pre-locality. For each $g \in \mathcal{L}$ define S_g to be the set of all $x \in \mathbf{D}(g) \cap S$ such that $x^g \in S$. Then:*

- (a) $S_g \in \Delta$. In particular, S_g is a subgroup of S .
- (b) The conjugation map $c_g : S_g \rightarrow (S_g)^g$ is an isomorphism of groups, and $S_{g^{-1}} = (S_g)^g$.
- (c) P^g is defined and is a subgroup of S , for every subgroup P of S_g . In particular, $P^g \in \Delta$ for any $P \in \Delta$ with $P \leq S_g$.

Proof. Fix $g \in \mathcal{L}$. Then the word (g) of length 1 is in \mathbf{D} by 1.1(2), and since $\mathbf{D} = \mathbf{D}_{\Delta}$ by (O1) there exists $X \in \Delta$ such that $Y := X^g \in \Delta$. Let $a \in S_g$ and set $b = a^g$. Then X^a and X^b are subgroups of S (as $a, b \in S$), so X^a and Y^b are in Δ by (O2). Then $(a^{-1}, g, b) \in \mathbf{D}$ via X^a , so also $(g, b) \in \mathbf{D}$. Also $(a, g) \in \mathbf{D}$ via $X^{a^{-1}}$. Since $g^{-1}ag = b$ we get $ag = gb$ by cancellation, and hence

$$a^{-1}gb = a^{-1}(gb) = a^{-1}(ag) = (a^{-1}a)g = g$$

by **D**-associativity. Since $a^{-1}gb$ conjugates X^a to Y^b , we draw the following conclusion.

(1) $X^a \leq S_g$ and $(X^a)^g \in \Delta$ for all $a \in S_g$.

Now let $c, d \in S_g$. Then (1) shows that both X^c and X^{cd} are members of Δ which are conjugated to members of Δ by g . Setting $w = (g^{-1}, c, g, g^{-1}, d, g)$, we conclude (by following X^g along the chain of conjugations given by w) that $w \in \mathbf{D}$ via X^g . Then **D**-associativity yields

$$(2) \quad \Pi(w) = (cd)^g = c^g d^g.$$

Since c^g and d^g are in S , we conclude that $cd \in S_g$. Since S_g is closed under inversion by 1.6(b), S_g is a subgroup of S . As $X \leq S_g \leq S$, where X and S are in Δ , (02) now yields $S_g \in \Delta$. Thus (a) holds.

Since $c_{g^{-1}} = (c_g)^{-1}$ by 1.6(c), it follows that $S_{g^{-1}} = (S_g)^g$. Points (b) and (c) are then immediate from (a) and 2.3(b). \square

It will be convenient to extend the notation introduced preceding 1.5. Thus, for \mathcal{L} a partial group and non-empty word $w = (g_1, \dots, g_n) \in \mathbf{W}(\mathcal{L})$, write $\mathbf{D}(w)$ for the set of all $x \in \mathcal{L}$ such that $x := x_0 \in \mathbf{D}(g_1)$, $x_1 := (x_0)^{g_1} \in \mathbf{D}(g_2)$, and so on until $y := x_n = (x_{n-1})^{g_n}$. We then write

$$c_w : \mathbf{D}(w) \rightarrow \mathcal{L}$$

for the mapping $x \mapsto y$, and call this mapping *conjugation by w* . Thus, $\mathbf{D}(w)$ is the largest subset of \mathcal{L} on which the composition

$$c_{g_1} \circ \dots \circ c_{g_n}$$

is defined. If w is the empty word define $\mathbf{D}(w)$ to be \mathcal{L} , and define c_w to be the identity map on \mathcal{L} . For any subset or element X of $\mathbf{D}(w)$, write X^w for the image of X under c_w .

For each $w \in \mathbf{W}(\mathcal{L})$ define S_w to be the set of all $x \in \mathbf{D}(w)$ such that, in the preceding description of c_w , we have $x_i \in S$ for all i ($0 \leq i \leq n$). In other words: S_w is the set of all $x \in S$ such that x is conjugated consecutively into S by the sequence $(c_{g_1}, \dots, c_{g_n})$ of conjugation maps. Notice that $(S_w)^w = S_{w^{-1}}$. The following lemma is a straightforward consequence of these definitions.

Lemma 2.9. *Let (\mathcal{L}, Δ, S) be a pre-locality, and let $u, v \in \mathbf{W}(\mathcal{L})$. Then:*

- (a) $S_{u \circ v} = ((S_u)^u \cap S_v)^{u^{-1}}$, and
- (b) $S_u \cap S_v = S_{u \circ u^{-1} \circ v}$.

Corollary 2.10. *Let (\mathcal{L}, Δ, S) be a pre-locality, and let $w \in \mathbf{W}(\mathcal{L})$. Then S_w is a subgroup of S for all $w \in \mathbf{W}(\mathcal{L})$, and $S_w \in \Delta$ if and only if $w \in \mathbf{D}$.*

Proof. If the length of w is at most 1 then S_w is a subgroup of S by 2.8(a). This provides the basis for an induction which is completed by 2.9(a). If $S_w \in \Delta$ then $w \in \mathbf{D}$ by the condition (O1) in definition 2.1. Conversely, if $w \in \mathbf{D}$ then (O1) says that S_w contains a member of Δ , and hence $S_w \in \Delta$ by (O2). \square

Definition 2.11. Let (\mathcal{L}, Δ, S) be a pre-locality. For each subgroup X of S set

$$X^\star = \bigcap \{S_w \mid w \in \mathbf{W}(\mathcal{L}), X \leq S_w\},$$

and set

$$\Omega := \Omega_S(\mathcal{L}) = \{X^\star \mid X \leq S\}.$$

From now on (\mathcal{L}, Δ, S) will be a pre-locality, and we shall write Ω for $\Omega_S(\mathcal{L})$.

Lemma 2.12. Let (\mathcal{L}, Δ, S) be a pre-locality, and let X, Y be subgroups of S .

- (a) $X^\star = (X^\star)^\star$.
- (b) If $X \leq Y$ then $X^\star \leq Y^\star$.
- (c) $\langle X^\star, Y^\star \rangle \leq \langle X, Y \rangle^\star$.

Proof. Points (a) and (b) are immediate from definition 2.11. Point (c) is an application of (b), with $\langle X, Y \rangle$ in the role of Y . \square

Lemma 2.13. Let (\mathcal{L}, Δ, S) be a pre-locality, let $u \in \mathbf{W}(\mathcal{L})$, and let X be a subgroup of S_u . Then $(X^\star)^u = (X^u)^\star$.

Proof. Set $\mathbf{W} = \mathbf{W}(\mathcal{L})$. Then

$$X^\star = \bigcap \{S_w \mid w \in \mathbf{W}, X \leq S_w\} = \bigcap \{(S_u \cap S_w) \mid w \in \mathbf{W}, X \leq S_w\} \leq S_u.$$

Then also

$$(X^\star)^u = \bigcap \{(S_u \cap S_w)^u \mid w \in \mathbf{W}, X \leq S_w\}.$$

Let $w \in \mathbf{W}(\mathcal{L})$ with $X \leq S_w$. Then 2.9(a) yields

$$(S_u \cap S_w)^u = ((S_{u^{-1}})^{u^{-1}} \cap S_w)^u = S_{u^{-1} \circ w},$$

and so

$$\begin{aligned} (X^\star)^u &= \bigcap \{S_{u^{-1} \circ w} \mid w \in \mathbf{W}, X \leq S_w\} \\ &\leq \bigcap \{S_{w'} \mid w' \in \mathbf{W}, X^u \leq S_{w'}\} \leq (X^u)^\star. \end{aligned}$$

We may apply this result with X^u in the role of X and with u^{-1} in place of u , obtaining $(X^u)^\star \leq S_{u^{-1}}$ and

$$((X^u)^\star)^{u^{-1}} \leq ((X^u)^{u^{-1}})^\star = X^\star.$$

Conjugation by u then yields $(X^u)^\star \leq (X^\star)^u$, and we thus have the two inclusions required for equality. \square

Regard Ω as a poset via the partial order given by inclusion, and define $\dim(\Omega)$ to be the supremum (if such a supremum exists) of the numbers d for which there exists a chain

$$(*) \quad X_0 < \cdots < X_d$$

of proper inclusions in Ω . Similarly, for any $X \in \Omega$ write $\dim_\Omega(X)$ for the supremum (if it exists) of the numbers d for which there exists a chain $(*)$ with $X_d = X$. The pre-locality (\mathcal{L}, Δ, S) will be said to be *finite-dimensional* if $\dim(\Omega) < \infty$.

Proposition 2.14. *Let (\mathcal{L}, Δ, S) be a finite-dimensional pre-locality, and set $\Omega = \Omega_S(\mathcal{L})$. Then $\Omega = \{S_w \mid w \in \mathbf{W}(\mathcal{L})\}$, and Ω is closed with respect to arbitrary intersections.*

Proof. By the definition of X^\star for X a subgroup of S , it suffices to show that Ω is closed with respect to arbitrary intersections. Such is the case, by an application of 2.9(b), and by finite-dimensionality. \square

We now extend the notion of dimension to all subgroups of S , by the formula:

$$\dim(X) = \dim(X^\star).$$

Corollary 2.15. *Let (\mathcal{L}, Δ, S) be a finite-dimensional pre-locality, and let X and Y be subgroups of S . Then the following hold.*

- (a) $\dim(X) = \dim(X^w)$ for all $w \in \mathbf{W}(\mathcal{L})$ with $X \leq S_w$.
- (b) If $X \leq Y \leq S$ then $\dim(X) \leq \dim(Y)$.
- (c) If $X \leq Y \leq S$ with $X = X^\star$, then $\dim(X) = \dim(Y)$ if and only if $X = Y$.

Proof. Let w be as in (a), and let $\sigma = (X_0 \leq \dots \leq X_d = X^\star)$ be a chain of strict inclusions in Ω , of length $d = \dim(X)$. As $X^\star \leq S_w$ by 2.13, conjugation by w sends σ to a chain of strict inclusions terminating in $(X^\star)^w$. As $(X^\star)^w = (X^w)^\star$ (by 2.13), we obtain (a). Point (b) is immediate from 2.12(b). Now suppose that $X = X^\star \leq Y$. Then $X \leq Y^\star$, and $\dim(X) = \dim(Y)$ if and only if $X = Y^\star$. As $X \leq Y \leq Y^\star$, (c) follows. \square

Lemma 2.16. *Let (\mathcal{L}, Δ, S) and (\mathcal{H}, Γ, R) be pre-localities, and let $\beta : \mathcal{H} \rightarrow \mathcal{L}$ be an injective homomorphism of partial groups such that $R\beta \leq S$. For any subgroup X of R , write $X^\bullet = \bigcap \{R_w \mid w \in \mathbf{W}(\mathcal{H}), P \leq R_w\}$.*

- (a) *There is an injective homomorphism $\Omega_R(\mathcal{H}) \rightarrow \Omega_S(\mathcal{L})$ of posets, given by $X \mapsto (X\beta)^\star$.*
- (b) *Assume that \mathcal{L} is finite-dimensional. Then $\dim(\Omega_R(\mathcal{H})) \leq \dim(\Omega_S(\mathcal{L}))$, and if equality holds then $(\mathbf{1}^\bullet)\beta \leq \mathbf{1}^\star$ and $(R\beta)^\star = S$.*

Proof. In proving (a) and (b) we may identify \mathcal{H} with the image of β , and so we may assume that β is an inclusion map.

By 2.12(b), the map $X \mapsto X^\star$ is a homomorphism $\Omega_R(\mathcal{H}) \rightarrow \Omega_S(\mathcal{L})$ of posets. In order to complete the proof of (a), the key observation is that $X^\star \cap R \leq X^\bullet$ for any subgroup X of R . Thus $X^\star \cap R = X$ if $X \in \Omega_R(\mathcal{H})$. If also $Y \in \Omega_R(\mathcal{H})$ and $X^\star = Y^\star$ then $X = R \cap X^\star = R \cap Y^\star = Y$, and so (a) holds.

Assume now that \mathcal{L} is finite-dimensional, and let $\mathbf{1}^\bullet < X_1 < \dots < X_d$ be a chain of proper inclusions in $\Omega_R(\mathcal{H})$. Since $\mathbf{1} \leq \mathbf{1}^\bullet \leq X$ for all $X \in \Omega_R(\mathcal{H})$, we obtain

$$\mathbf{1}^\star \leq (\mathbf{1}^\bullet)^\star < (X_1)^\star < \dots < (X_d)^\star \leq S$$

which is a chain of length greater than d unless $\mathbf{1}^\star = (\mathbf{1}^\bullet)^\star$ and $(X_d)^\star = S$. This proves (b). \square

Proposition 2.17. *Let (\mathcal{L}, Δ, S) be a finite-dimensional pre-locality and let H be a subgroup of \mathcal{L} . Then there exists $P \in \Delta \cap \Omega$ such that $H \leq N_{\mathcal{L}}(P)$. Moreover, P may be chosen so that $P = S_u$ for some $u \in \mathbf{W}(H)$, and then P is the unique largest subgroup of S normalized by H .*

Proof. Among all $u \in \mathbf{W}(H)$, choose u so that $\dim(S_u)$ is as small as possible, and set $P = S_u$. As H is a subgroup of \mathcal{L} we have $u \in \mathbf{D}$. Thus $P \in \Delta \cap \Omega$, and P contains every subgroup Q of S such that $H \leq N_{\mathcal{L}}(Q)$. Let v be an arbitrary element of $\mathbf{W}(H)$, and set $w = u \circ u^{-1} \circ v$. Then $S_w = S_u \cap S_v$ by 2.11(b), and $S_w \leq S_u$. Then $\dim(S_w) = \dim(S_u)$ by the minimality of $\dim(S_u)$, and so $S_w = P$ by 2.15(c). Thus:

$$P = \bigcap \{S_v \mid v \in \mathbf{W}(H)\}.$$

Now let $h \in H$. Then $P = S_{(h, h^{-1}) \circ u}$ and $P = S_{(h^{-1}, h) \circ u}$, whereas evidently $P^h = S_{(h^{-1}, h) \circ u}$. Thus $P = P^h$, and $H \leq N_{\mathcal{L}}(P)$ by 2.12. Replacing P with P^* yields $H \leq N_{\mathcal{L}}(P)$. If also $Q \in \Delta \cap \Omega$ with $H \leq N_{\mathcal{L}}(Q)$ then $Q \leq S_u = P$. \square

Section 3: Localities

Here is the main definition.

Definition 3.1. A *locality* is a finite-dimensional pre-locality (\mathcal{L}, Δ, S) satisfying the following two conditions.

- (L1) S is a p -group for some prime p , and S is maximal among the p -subgroups of \mathcal{L} .
- (L2) Each subgroup of \mathcal{L} is locally finite and countable.

Remarks. (1) By a p -group we mean a group all of whose elements are of (finite) order a power of p . The p -subgroups of a partial group form a poset via inclusion, and it is with respect to this poset that we say that a given p -group is maximal.

(2) Notice that by 2.17, (L2) is equivalent to the requirement that each of the groups $N_{\mathcal{L}}(P)$, for $P \in \Delta$, be locally finite and countable. Another equivalent formulation of (L2) (and one which goes some way towards explaining the countability requirement) is that each subgroup of \mathcal{L} is a nested union of finite groups. Since we shall make use of this last formulation from time to time, it will be convenient to establish the following terminology: a *framing* of a group G (by finite subgroups) is a collection $\{G_n\}$ of finite subgroups of G , indexed by the non-negative integers, such that $G_n \leq G_{n+1}$ for all n , and such that $G = \bigcup \{G_n\}$. We shall write also $G = \lim \{G_n\}$ if $\{G_n\}$ is a framing of G .

(3) The results 2.12 through 2.15 from the preceding section, which concern formal properties of $\Omega_S(\mathcal{L})$ and of the operation $V \mapsto V^*$ on subgroups of S will be employed so often, and are (we believe) so natural, that explicit reference to them will in general be omitted.

(4) It will often be convenient, within discussions involving p -groups, to adopt the notation $P < Q$ to indicate that P is a proper subgroup of Q .

Lemma 3.2. *Let (\mathcal{L}, Δ, S) be a locality, let V be a subgroup of S , and let X be a p -subgroup of \mathcal{L} such that $V < X$. Then $V < N_X(V^*)$, and if $V = V^* \cap X$ then $V < N_X(V)$. In particular, we have $V < N_X(V)$ if $V \in \Omega$.*

Proof. Let $\{X_n\}$ be a framing of X by finite subgroups, and set $V_n = V \cap X_n$. For n sufficiently large we have $(V_n)^* = V^*$ (by finite-dimensionality) and $V_n < X_n$. As X_n is a finite p -group there exists $x \in N_{X_n}(V_n)$ with $x \notin V_n$. Then $x \notin V$, while $x \in N_X(V^*)$ by 2.13. Thus $V < N_X(V^*)$, and if $V = V^* \cap X$ then $V < N_X(V)$. \square

Corollary 3.3. *Let $V \in \Omega$, and let $w \in \mathbf{W}(\mathcal{L})$ with $V \leq S_w$. Set $P = N_{S_w}(V)$, set $Q = N_S(V)$, and assume that $P < Q$. Then $\dim(P) < \dim(N_Q(P))$.*

Proof. We have $P^* \leq S_w$, so $P^* \cap Q \leq N_{S_w}(V) = P$. As $P < Q$ we then have $P < N_Q(P)$ by 3.2. Suppose that $P^* = N_Q(P)^*$. Then

$$P < N_Q(P) \leq N_Q(P)^* \cap Q = P^* \cap Q = P,$$

for a contradiction. Thus $P^* < N_Q(P)^*$, and so $\dim(P) < \dim(N_Q(P))$. \square

For each $w \in \mathbf{W}(\mathcal{L})$ there is a conjugation map $c_w : X \rightarrow Y$, for any subgroup X of S_w , and for any subgroup Y of S containing X^w (cf. the discussion following 2.8).

Definition 3.4. Let (\mathcal{L}, Δ, S) be a locality. The *fusion system* $\mathcal{F} := \mathcal{F}_S(\mathcal{L})$ is the category whose objects are the subgroups of S , and whose morphisms are the conjugation maps

$$c_w : X \rightarrow Y \quad (\text{where } w \in \mathbf{W}(\mathcal{L}), X \leq S_w, \text{ and } X^w \leq Y).$$

The fusion system $\mathcal{F}_S(\mathcal{L})$ will be the focus of Part II of this series, and it will play an important role in the remaining Parts. In this Part I, the only reason for introducing it is for the sake of some convenient terminology and notation, as follows.

- If $X \leq S_w$ then X^w is an \mathcal{F} -conjugate of X , and the set of all \mathcal{F} -conjugates of X is denoted $X^{\mathcal{F}}$.
- A subgroup X of S is *fully normalized* in \mathcal{F} if $\dim(N_S(X)) \geq \dim(N_S(Y))$ for every \mathcal{F} -conjugate Y of X . Notice that since $\dim(\Omega)$ is finite, every subgroup of S has a fully normalized \mathcal{F} -conjugate.
- A collection Γ of subgroups of S is \mathcal{F} -invariant if Γ is closed with respect to \mathcal{F} -isomorphisms; and Γ is \mathcal{F} -closed if Γ is non-empty and closed with respect to all \mathcal{F} -homomorphisms ($X \in \Gamma$, $w \in \mathbf{W}(\mathcal{L})$ with $X \leq S_w$, and $X^w \leq Y \leq S$ implies $Y \in \Gamma$). For example, Δ is \mathcal{F} -closed.

From this point forth, (\mathcal{L}, Δ, S) will be a fixed locality. We write \mathcal{F} for $\mathcal{F}_S(\mathcal{L})$, and we say also that \mathcal{L} is a locality *on* \mathcal{F} . Write Ω for $\Omega_S(\mathcal{L})$. For any subgroup X of S write $\dim(X)$ for $\dim_{\Omega}(X^*)$.

The following result is immediate from 2.3(c).

Lemma 3.5. For all $P, Q \in \Delta$, $\text{Hom}_{\mathcal{F}}(P, Q)$ is the set of conjugation maps $c_g : P \rightarrow Q$ such that $g \in \mathcal{L}$, $P \leq S_g$, and $P^g \leq Q$. In particular, for each $P \in \Delta$ there exists $g \in \mathcal{L}$ such that P^g is fully normalized in \mathcal{F} . \square

Lemma 3.6. Set $O_p(\mathcal{L}) = \mathbf{1}^*$. Then $O_p(\mathcal{L})$ is a normal p -subgroup of \mathcal{L} , and contains every normal p -subgroup of \mathcal{L} . Moreover, if \mathcal{L} is a group then $O_p(\mathcal{L}) = \bigcap \text{Syl}_p(\mathcal{L})$.

Proof. Let X be a normal p -subgroup of \mathcal{L} . Then SX is a p -group, and so $X \leq S$ by the maximality of S among the p -subgroups of \mathcal{L} . Then $X \leq S_w$ for all $w \in \mathbf{W}(\mathcal{L})$, and thus $X \leq \mathbf{1}^*$. Since $\mathbf{1}^* \trianglelefteq \mathcal{L}$ by 2.13, we have the desired characterization of $O_p(\mathcal{L})$. If \mathcal{L} is a group then $\text{Syl}_p(\mathcal{L})$ is a conjugacy class of subgroups of \mathcal{L} , and so $O_p(\mathcal{L}) = \bigcap \text{Syl}_p(\mathcal{L})$ in that case. \square

Definition 3.7. A p -subgroup R of a partial group \mathcal{G} is a *Sylow p -subgroup* of \mathcal{G} if

- (1) R is a maximal p -subgroup of \mathcal{G} and,
- (2) for each p -subgroup P of \mathcal{G} there exists $g \in \mathcal{G}$ such that P^g is a subgroup of R .

We write $\text{Syl}_p(\mathcal{G})$ for the set (possibly empty) of all Sylow p -subgroups of \mathcal{G} .

Proposition 3.8. Let (\mathcal{L}, Δ, S) be a locality. Then $S \in \text{Syl}_p(\mathcal{L})$.

Proof. Set $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$ and $\Omega = \Omega_S(\mathcal{L})$. Notice that $N_{\mathcal{L}}(V)$ is a partial subgroup of \mathcal{L} for each subgroup V of S . We first show:

- (1) Let $V \leq S$ with V fully normalized in \mathcal{F} , and assume that for each p -subgroup X of $N_{\mathcal{L}}(V)$ with $N_S(V) \trianglelefteq X$, there exists $g \in \mathcal{L}$ such that $V^g \leq S$. Then $N_S(V)$ is a maximal p -subgroup of $N_{\mathcal{L}}(V)$.

Indeed, set $R = N_S(V)$, let X be a p -subgroup of $N_{\mathcal{L}}(V)$ containing R , and set $D = N_X(R)$. By assumption there exists $g \in \mathcal{L}$ with $D^g \leq S$, and so $R^g \leq D^g \leq N_S(V^g)$. As V is fully normalized in \mathcal{F} it follows that $\dim(R^g) = \dim(D^g) = \dim(N_S(V^g))$. Then $D^g \leq (R^g)^* = (R^*)^g$. Then $D \leq R^*$, so $D \leq S$, and $D \leq S \cap N_{\mathcal{L}}(V) = R$. Thus $R = N_X(D)$, and so $R = X$ by 3.2. This completes the proof of (1).

Among all counterexamples to 3.8, choose \mathcal{L} so that $\dim(\Omega)$ is as small as possible. By 2.17, for each p -subgroup X of \mathcal{L} there exists a unique largest $P_X \in \Delta \cap \Omega$ with $X \leq N_{\mathcal{L}}(P_X)$. Then XP_X is a p -group. Let \mathbf{X} be the set of all p -subgroups X of \mathcal{L} such that $P_X \leq X$ and such that there exists no $g \in \mathcal{L}$ with $X^g \leq S$. Thus \mathbf{X} is non-empty. Choose $X \in \mathbf{X}$ so that $\dim(P_X)$ is as large as possible, and set $P = P_X$.

Let $w \in \mathbf{W}(\mathcal{L})$ such that $P \leq S_w$. Then 2.3 shows that X^w is defined and is a p -subgroup of $N_{\mathcal{L}}(P^w)$. If there exists $g \in \mathcal{L}$ such that $(X^w)^g \leq S$ then $X^f \leq S$, where $f = \Pi(w \circ (g))$, and where $w \circ (g) \in \mathbf{D}$ via P . Thus, we are free to replace X by any X^g such that $P^g \leq S$, and we may therefore assume that X has been chosen so that P is fully normalized in \mathcal{F} . Set $R = N_S(P)$, and $H = N_{\mathcal{L}}(P)$.

If $P = S$ then SX is a p -subgroup of \mathcal{L} , and then $X \leq S$ by (L1), contrary to $X \in \mathbf{X}$. Thus $P < S$, and since $P \in \Omega$ we obtain $P < R$ from 3.2. Then $\dim(P) < \dim(R)$. Now let X' be a p -subgroup of H such that $R \trianglelefteq X'$. Then $R \leq P_{X'}$, so the maximality of $\dim(P)$ in the choice of X yields $X' \notin \mathbf{X}$. That is, X' is conjugate in \mathcal{L} to a subgroup of S ; and then (1) implies that R is a maximal p -subgroup of H . Then H may be viewed

as a locality (H, Γ, R) , in which Γ is the set of all subgroups of R . As $X \in \mathbf{X}$, X is not conjugate in H to a subgroup of R , and thus (H, Γ, R) is a counter-example to the proposition. As $\dim(\Omega_R(H)) \leq \dim(\Omega)$ by 2.16, we may then assume that $H = \mathcal{L}$. Thus $R = S$, and $P = O_p(H)$ by 3.5.

Let \mathbf{M} be the set of all maximal p -subgroups of H , and let $\widetilde{\mathbf{M}}$ be the set of all $S' \in \mathbf{M}$ such that there exists a framing $\{H_n\}$ of H by finite subgroups with $S' \cap H_n \in \text{Syl}_p(H_n)$ for all n (see the remark following 3.1). Notice that if $S \notin \widetilde{\mathbf{M}}$ then no member of $\widetilde{\mathbf{M}}$ is conjugate to S in H . Thus we may choose $S' \in \mathbf{M}$ such that S' is not conjugate to S , and we may assume that either S or S' is in $\widetilde{\mathbf{M}}$. Among all such S' , choose S' so that $\dim(S \cap S')$ is as large as possible, and set $S_n = S \cap H_n$ and $S'_n = S' \cap H_n$. If $S = P$ then $S \trianglelefteq H$, and then (L1) implies that $S \in \text{Syl}_p(H)$. As H is a counterexample to the proposition, we conclude that $P < S$. Thus there exists n with $S_n \not\leq P$. Also $S' \not\leq P$ since $S' \not\leq S$. Since one of S_n or S'_n is conjugate into the other in H_n , we conclude that $P < S \cap S'$. Thus $\dim(S \cap S') > \dim(P)$.

Set $Y = (S \cap S')^*$. Then there exists $x \in G$ with Y^x fully normalized in \mathcal{F} . Upon replacing S' with $(S')^x$, we may therefore assume that Y is fully normalized in \mathcal{F} . Then (1) implies that $N_S(Y)$ is a maximal p -subgroup of $N_H(Y)$. Here $\dim(\Omega_{N_S(Y)}(N_H(Y))) < \dim(\Omega)$ by 2.16(b), so the locality $N_H(Y)$ is not a counterexample, and $N_S(Y)$ is a Sylow p -subgroup of $N_H(Y)$. Set $T = N_{S'}(S \cap S')$. Then $T \leq N_H(Y)$, so there exists $h \in N_H(Y)$ with $X^h \leq N_S(Y)$. The maximality condition on $\dim(S \cap S')$ then yields $\dim(S \cap S') = \dim(S \cap (S')^h)$. Then, since $(S \cap S')^h \leq T^h \leq S \cap (S')^h$, we conclude that

$$Y^h = (T^h)^* = (T^*)^h \geq T^h.$$

Conjugation by h^{-1} now yields $Y \geq T$, and so $T = S \cap S'$. We now appeal to 3.2, and conclude that $S \cap S' = S'$. The maximality of S' then yields $S' = S$, thereby providing a contradiction and completing the proof. \square

The following corollary is immediate from 3.8.

Corollary 3.9. *Let $V \leq S$ with V fully normalized in \mathcal{F} . Then $N_S(V)$ is a maximal p -subgroup of the partial group $N_{\mathcal{L}}(V)$. \square*

Corollary 3.10. *Let $P \in \Delta$. Then $N_S(P) \in \text{Syl}_p(N_{\mathcal{L}}(P))$ if and only if P is fully normalized in \mathcal{F} .*

Proof. Set $R = N_S(P)$ and $H = N_{\mathcal{L}}(P)$. Suppose that P is fully normalized in \mathcal{F} . Then R is a maximal p -subgroup of H by 3.7, and then (H, Γ, R) is a locality, where Γ is the set of all subgroups of R . Then $R \in \text{Syl}_p(H)$ by 3.7. For the converse; suppose that $R \in \text{Syl}_p(H)$ and let P' be an \mathcal{F} -conjugate of P . Then $P' = P^g$ for some $g \in \mathcal{L}$ by 2.3(c), and conjugation by g is an isomorphism $H \rightarrow N_{\mathcal{L}}(P')$ by 2.3(b). Set $T = N_S(P')^{g^{-1}}$. Then T is a p -subgroup of H , and so there exists $h \in H$ with $(T^g)^h \leq R$. As $(g^{-1}, h) \in \mathbf{D}$ via P' , we have $T^{gh} \leq R$. Thus $\dim(T) \leq \dim(R)$, and so P is fully normalized in \mathcal{F} . \square

Corollary 3.11. *Let H be a subgroup of \mathcal{L} . Then $\text{Syl}_p(H) \neq \emptyset$, and $\text{Syl}_p(H)$ is the set of maximal p -subgroups of H . Moreover, for any $R \in \text{Syl}_p(H)$ there exists a framing $\{H_n\}$ of H by finite subgroups such that $R \cap H_n \in \text{Syl}_p(H_n)$ for all n .*

Proof. Let R be a maximal p -subgroup of H and let Γ be the set of all subgroups of R . Then (H, Γ, R) is a locality, and so $R \in \text{Syl}_p(H)$. By the remark following 3.1 one may choose R so that $R \cap H_n \in \text{Syl}_p(H_n)$ for all n . One observes that for any $h \in H$, $\{(H_n)^h\}$ is again a framing of H . Since $\text{Syl}_p(H)$ forms a single H -conjugacy class we then have the desired “compatibility” with a suitable framing, for any Sylow p -subgroup of H . \square

The proofs of the three parts of the following result are essentially the same as the corresponding (very elementary) proofs for finite groups, and they are therefore omitted.

Lemma 3.12. *Let H be a subgroup of \mathcal{L} , let $K \trianglelefteq H$ be a normal subgroup of H , let R be a Sylow p -subgroup of H , and set $T = R \cap K$. Then the following hold.*

- (a) $T \in \text{Syl}_p(K)$ and $G = N_G(T)K$.
- (b) For any subgroup Q of R with $T \leq Q$, we have $Q \in \text{Syl}_p(KQ)$.
- (c) $RK/K \in \text{Syl}_p(H/K)$.

We end this section with three results on localities formed within \mathcal{L} .

Lemma 3.13. *Let $T \trianglelefteq S$ be a normal subgroup of S , and set $\mathcal{L}_T = N_{\mathcal{L}}(T)$. Then $(\mathcal{L}_T, \Delta, S)$ is a locality.*

Proof. That \mathcal{L}_T is a partial subgroup of \mathcal{L} follows from 2.3(c). That (\mathcal{L}_T, Δ) is objective is a consequence of 2.10. The remaining points of verification are inherited from \mathcal{L} in a straightforward way. \square

Recall from definition 3.4 the notion of an \mathcal{F} -closed collection Γ of subgroups of S . The reader should have no difficulty in working through the steps which verify the following result.

Lemma 3.14. *Let Γ be an \mathcal{F} -closed subset of Δ . Set*

$$\mathcal{L} \upharpoonright_{\Gamma} = \{g \in \mathcal{L} \mid S_g \in \Gamma\}, \quad \text{and} \quad \mathbf{D} \upharpoonright_{\Gamma} = \{w \in \mathbf{W}(\mathcal{L}) \mid S_w \in \Gamma\}.$$

Then $\mathcal{L} \upharpoonright_{\Gamma}$ is a partial group by restriction of the product in \mathcal{L} to $\mathbf{D} \upharpoonright_{\Gamma}$, and by restriction of the inversion of the inversion in \mathcal{L} . Moreover, $(\mathcal{L} \upharpoonright_{\Gamma}, \Gamma, S)$ is a locality (to be called the restriction of \mathcal{L} to Γ). \square

Lemma 3.15. *Let (\mathcal{L}, Δ, S) be a locality, and let $\tilde{\Delta}$ be the collection of all subgroups $P \leq S$ such that $P^* \in \Delta$. Then $(\mathcal{L}, \tilde{\Delta}, S)$ is a locality.*

Proof. It follows from 2.13 that $\tilde{\Delta}$ is \mathcal{F} -invariant, and then from 2.12(b) that $\tilde{\Delta}$ is \mathcal{F} -closed. Thus $(\mathcal{L}, \tilde{\Delta})$ satisfies condition (O2) in definition 2.1. Also, 2.13 shows that condition (O1) is satisfied, and so $(\mathcal{L}, \tilde{\Delta})$ is an objective partial group. The conditions that $(\mathcal{L}, \tilde{\Delta}, S)$ must then fulfill in order to be a locality are given by the partial group structure of \mathcal{L} , and so the lemma holds. \square

Section 4: Partial normal subgroups

Throughout this section we fix a locality (\mathcal{L}, Δ, S) and a partial normal subgroup $\mathcal{N} \trianglelefteq \mathcal{L}$. Recall that this means that \mathcal{N} is a partial subgroup of \mathcal{L} and that $\Pi(g^{-1}, x, g) \in \mathcal{N}$ for all $x \in \mathcal{N}$ and all $g \in \mathcal{L}$ for which $(g^{-1}, x, g) \in \mathbf{D}$. Set $T = S \cap \mathcal{N}$.

Recall from 3.4 that there is a fusion system $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$ associated with \mathcal{L} . A subgroup R of S is *strongly closed* in \mathcal{F} if $x^w \in R$ whenever $w \in \mathbf{W}(\mathcal{L})$ with $x \in R \cap S_w$.

Lemma 4.1.

- (a) T is strongly closed in \mathcal{F} .
- (b) Let $x \in \mathcal{N}$ and let P be a subgroup of S_x . Then $PT = P^xT$.
- (c) T is maximal in the poset of p -subgroups of \mathcal{N} .

Proof. (a) Let $g \in \mathcal{L}$ and let $t \in S_g \cap T$. Then $t^g \in S$, and $t^g \in \mathcal{N}$ as $\mathcal{N} \trianglelefteq \mathcal{L}$. Thus $t^g \in T$. Iteration of such conjugation maps shows that $t^w \in T$ for all $w \in \mathbf{W}(\mathcal{L})$ with $t \in S_w$.

(b) Let $a \in P$. Then $(P^x)^a \leq S$ and $P^a = P$. Setting $w = (a^{-1}, x^{-1}, a, x)$ we then have $w \in \mathbf{D}$ via P^{xa} . Now $\Pi(w) = a^{-1}a^x \in S$, while also $\Pi(w) = (x^{-1})^a x \in \mathcal{N}$, and so $\Pi(w) \in T$. Then $a^x \in aT$, and we have thus shown that $P^x \leq PT$. Then $P^xT \leq PT$. The equality $P^xT = PT$ follows via symmetry, with x^{-1} and P^x in place of x and P .

(c) Let X be a p -subgroup of \mathcal{N} containing T . By 3.8 there exists $g \in \mathcal{L}$ with $X^g \leq S$, and then $X^g \leq S \cap \mathcal{N} = T$. Set $P = S_g$ and $Q = P^g$. Then $X^g \leq Q$ and conjugation by g^{-1} yields $X \leq P$. Thus $X \leq P \cap \mathcal{N} = T$, and so $X = T$. \square

Lemma 4.2. Let $x, y \in \mathcal{N}$ and let $f \in N_{\mathcal{L}}(T)$.

- (a) If $(x, f) \in \mathbf{D}$ then $(f, f^{-1}, x, f) \in \mathbf{D}$, $xf = fx^f$, and $S_{(x, f)} = S_{(f, x^f)} = S_x \cap S_f$.
- (b) If $(f, y) \in \mathbf{D}$ then $(f, y, f^{-1}, f) \in \mathbf{D}$, $fy = y^{f^{-1}}f$, and $S_{(f, y)} = S_{(y^{f^{-1}}, f)} = S_{y^{f^{-1}}} \cap S_f$.

Proof. (a): Set $Q = S_{(x, f)}$. Then $T \leq S_f$ by hypothesis. Since $Q^xT = QT$ by 4.1(b), we then have $Q \leq S_f$. Thus $Q \leq P := S_x \cap S_f$. But also $P^xT = PX$, so $P = Q$. Moreover, $(f, f^{-1}, x, f) \in \mathbf{D}$ via Q , and then $\Pi(f, f^{-1}, x, f) = xf = fx^f$. As $Q = S_f \cap S_x \leq S_{(f, f^{-1}, x, f)}$ we obtain $S_{(x, f)} = S_{(f, x^f)}$. Thus, (a) holds.

For point (b): Set $R = S_{(f, y)}$. Then $R^{fy}T = R^fT \leq S_{f^{-1}}$, so $(f, y, f^{-1}, f) \in \mathbf{D}$ via R , and $fy = y^{f^{-1}}f$. The remainder of (b) now follows as an application of (a) to $(y^{f^{-1}}, f)$. \square

Lemma 4.3. Let $w \in \mathbf{W}(N_{\mathcal{L}}(T)) \cap \mathbf{D}$, set $g = \Pi(w)$, and let $x, y \in \mathcal{N}$.

- (a) Suppose $(x) \circ w \in \mathbf{D}$ and set $P = S_{(x) \circ w}$. Then $u := w^{-1} \circ (x) \circ w \in \mathbf{D}$, and $S_u = P^g$.
- (b) Suppose that $w \circ (y) \in \mathbf{D}$ and set $Q = S_{w \circ (y)}$. Then $v := w \circ (y) \circ w^{-1} \in \mathbf{D}$, and $S_v = Q$.

Proof. We prove only (a), leaving it to the reader to supply a similar argument for (b). As $P^x T = PT$ by 4.1(b), and since both P^x and T are contained in S_w , we obtain $P \leq S_w$. Then $P^g \leq S$, and $P^g \leq S_u$. In particular $u \in \mathbf{D}$ via P^g . As $S_u \leq S_{w^{-1}}$, and $(S_u)^{g^{-1}} \leq P$, we obtain $S_u \leq P^g$. Thus $S_u = P^g$. \square

Lemma 4.4 (“Frattini Calculus”). *Let $w = (f_1, g_1, \dots, f_n, g_n) \in \mathbf{D}$ via P , with $f_i \in N_{\mathcal{L}}(T)$ and with $g_i \in \mathcal{N}$ for all i . Set $u_n = g_n$, and for all i with $1 \leq i < n$ set*

$$u_i = (f_n^{-1}, \dots, f_{i+1}^{-1}, g_i, f_{i+1}, \dots, f_n).$$

Then $u_i \in \mathbf{D}$ via P and, upon setting $\bar{g}_i = \Pi(u_i)$ and $w' = (f_1, \dots, f_n, \bar{g}_1, \dots, \bar{g}_n)$, we have $w' \in \mathbf{D}$, $S_w = S_{w'}$, and $\Pi(w) = \Pi(w')$. Thus:

$$(a) \quad \Pi(w) = (f_1 \cdots f_n)(\bar{g}_1 \cdots \bar{g}_n), \text{ where } \bar{g}_i = (g_i)^{f_{i+1} \cdots f_n}.$$

Moreover, we similarly have:

$$(b) \quad \Pi(w) = (\tilde{g}_1 \cdots \tilde{g}_n)(f_1 \cdots f_n) \text{ where } \tilde{g}_i = (g_i)^{(f_1 \cdots f_i)^{-1}}.$$

Proof. For the proof of (a), suppose that we are given an index k such that

$$w_k = (f_1, g_1, \dots, f_k, g_k) \circ (f_{k+1}, \dots, f_n) \circ (\bar{g}_{k+1}, \dots, \bar{g}_n) \in \mathbf{D},$$

with $P := S_w = S_{w_k}$, $\Pi(w) = \Pi(w_k)$, $u_i \in \mathbf{D}$ for all i with $i > k$, and where $\bar{g}_i = \Pi(u_i)$. For example, $k = n$ is such an index. If $k = 0$ then (a) holds. So assume $k > 0$. Set

$$v = (f_1, g_1, \dots, f_{k-1}, g_{k-1}, f_k).$$

Then 4.3(a) as applied to $(g_k) \circ (f_{k+1}, \dots, f_n)$ yields $u_k \in \mathbf{D}$ via Q , where Q is the image of P under conjugation by $\Pi(v)$. Then also $w_{k-1} \in \mathbf{D}$ via P , $S_w = S_{w_{k-1}}$, and 1.1(3) yields $\Pi(w_{k-1}) = \Pi(w_k)$. Iteration of this procedure yields (a). A similar procedure involving 4.3(b) yields (b). \square

Definition 4.5. Let $\mathcal{L} \circ \Delta$ be the set of all pairs $(f, P) \in \mathcal{L} \times \Delta$ such that $P \leq S_f$. Define a relation \uparrow on $\mathcal{L} \circ \Delta$ by $(f, P) \uparrow (g, Q)$ if there exist elements $x \in N_{\mathcal{N}}(P, Q)$ and $y \in N_{\mathcal{N}}(P^f, Q^g)$ such that $xg = fy$.

This relation may be indicated by means of a commutative square:

$$(*) \quad \begin{array}{ccc} Q & \xrightarrow{g} & Q^g \\ x \uparrow & & \uparrow y \\ P & \xrightarrow{f} & P^f \end{array}$$

of conjugation maps, labeled by the conjugating elements, and in which the horizontal arrows are isomorphisms and the vertical arrows are injective homomorphisms. The relation $(f, P) \uparrow (g, Q)$ may also be expressed by:

$$w := (x, g, y^{-1}, f^{-1}) \in \mathbf{D} \text{ via } P, \text{ and } \Pi(w) = \mathbf{1}.$$

It is easy to see that \uparrow is reflexive and transitive. We say that (f, P) is *maximal* in $\mathcal{L} \circ \Delta$ if $(f, P) \uparrow (g, Q)$ implies $\dim(P) = \dim(Q)$. As \mathcal{L} is finite-dimensional there exist maximal elements in $\mathcal{L} \circ \Delta$. Since $(f, P) \uparrow (f, S_f)$ for $(f, P) \in \mathcal{L} \circ \Delta$, it follows that $P = S_f$ for every maximal (f, P) . For this reason, we will say that f is \uparrow -*maximal* in \mathcal{L} (with respect to \mathcal{N}) if (f, S_f) is maximal in $\mathcal{L} \circ \Delta$.

Lemma 4.6. *Let $f \in \mathcal{L}$.*

- (a) *If $f \in N_{\mathcal{L}}(S)$ then f is \uparrow -maximal.*
- (b) *If f is \uparrow -maximal then so is f^{-1} .*
- (c) *If f is \uparrow -maximal and $(f, S_f) \uparrow (g, Q)$, then g is \uparrow -maximal and $Q = S_g$.*

Proof. Point (a) is immediate from definition 4.5. Now suppose that f is \uparrow -maximal, and let $g \in \mathcal{L}$ with $(f^{-1}, S_{f^{-1}}) \uparrow (g^{-1}, S_{g^{-1}})$. Since $S_{f^{-1}} = (S_f)^f$ and $S_{g^{-1}} = (S_g)^g$, there is a diagram

$$\begin{array}{ccc} (S_g)^g & \xrightarrow{g^{-1}} & S_g \\ x \uparrow & & \uparrow y \\ (S_f)^f & \xrightarrow{f^{-1}} & S_f \end{array}$$

as in definition 4.5, from which it is easy to read off the relation $(f, S_f) \uparrow (g, S_g)$. Then $\dim(S_f) = \dim(S_g)$ as f is \uparrow -maximal. As $S_{x^{-1}} = (S_x)^x$ for any $x \in \mathcal{L}$ we obtain also $\dim(S_{f^{-1}}) = \dim(S_{g^{-1}})$. Thus f^{-1} is \uparrow -maximal, and (b) holds. Point (c) is immediate from the transitivity of \uparrow . \square

Lemma 4.7. *Let $(g, Q), (h, R) \in \mathcal{L} \circ \Delta$ with $(g, Q) \uparrow (h, R)$, and suppose that $T \leq R$. Then there exists a unique $y \in \mathcal{N}$ with $g = yh$. Moreover:*

- (a) $Q^y \leq R$, and $Q \leq S_{(y,h)}$.
- (b) *If $N_T(Q^g) \in \text{Syl}_p(N_{\mathcal{N}}(Q^g))$, then $N_T(Q^y) \in \text{Syl}_p(N_{\mathcal{N}}(Q^y))$.*

Proof. By definition of the relation \uparrow , there exist elements $u \in N_{\mathcal{N}}(Q, R)$ and $v \in N_{\mathcal{N}}(Q^g, R^h)$ such that $(u, h, v^{-1}, g^{-1}) \in \mathbf{D}$ via Q , and such that $\Pi(w) = \mathbf{1}$.

$$\begin{array}{ccc} R & \xrightarrow{h} & R^h \\ u \uparrow & & \uparrow v \\ Q & \xrightarrow[g]{} & Q^g \end{array}$$

In particular, $uh = gv$. Since $T \leq R$, points (a) and (b) of 4.1 yield

$$T = T^h, Q^u T = QT \leq R, \text{ and } Q^g T = Q^{gv} T \leq R^h.$$

Then

$$w := (u, h, v^{-1}, h^{-1}) \in \mathbf{D} \quad \text{via} \quad (Q, Q^u, Q^{uh}, Q^{uhv^{-1}} = Q^g, Q^{gh^{-1}}).$$

Set $y = \Pi(w)$. Then $y = u(v^{-1})^{h^{-1}} \in N_{\mathcal{N}}(Q, R)$. By 1.1 both $(u, h, v^{-1}, h^{-1}, h)$ and (g, v, v^{-1}) are in \mathbf{D} , and hence $yh = uhv^{-1} = g$. This yields (a). The uniqueness of y is given by right cancellation.

Suppose now that $N_T(Q^g) \in \text{Syl}_p(N_{\mathcal{N}}(Q^g))$. As $N_T(Q^y)^h = N_T(Q^g)$, it follows from 2.3(b) that $N_T(Q^y) \in \text{Syl}_p(N_{\mathcal{N}}(Q^y))$. \square

Proposition 4.8. *Let $g \in \mathcal{L}$ and suppose that g is \uparrow -maximal with respect to \mathcal{N} . Then $T \leq S_g$.*

Proof. Set $P = S_g$ and $Q = P^g$. We first show:

- (1) Let $y \in N_{\mathcal{N}}(P, S)$. Then $(g, P) \uparrow (y^{-1}g, P^y)$, and $y^{-1}g$ is \uparrow -maximal.

Indeed, we have a commutative diagram

$$\begin{array}{ccc} P^y & \xrightarrow{y^{-1}g} & P^g \\ y \uparrow & & \uparrow 1 \\ P & \xrightarrow[g]{} & P^g \end{array}$$

as in 4.5, and then (1) is given by 4.6(c).

Suppose next that $N_T(P) \in \text{Syl}_p(N_{\mathcal{N}}(P))$. Then $N_T(P)^g \in \text{Syl}_p(N_{\mathcal{N}}(Q))$ by 2.3(b), and so there exists $x \in N_{\mathcal{N}}(Q)$ such that $N_T(Q) \leq (N_T(P)^g)^x$. Here $(g, x) \in \mathbf{D}$ via P , so $(N_T(P)^g)^x = N_T(P)^{gx}$, and then $(g, P) \uparrow (gx, N_T(P)P)$. As g is \uparrow -maximal, we conclude that $N_T(P) \leq P$. Then (L2), as applied to the inclusion $P \leq TP$, yields $T \leq P$. Thus:

- (2) If $T \not\leq P$ then $N_T(P) \notin \text{Syl}_p(N_{\mathcal{N}}(P))$.

We next show:

- (3) Suppose that there exists $y \in \mathcal{N}$ such that $P \leq S_y$ and such that $N_T(P^y) \in \text{Syl}_p(N_{\mathcal{N}}(P^y))$. Then $T \leq P$.

Indeed, under the hypothesis of (3) we have $(g, P) \uparrow (y^{-1}g, P^y)$ by (1). Then $(y^{-1}g, P^y)$ is \uparrow -maximal and $P^y = S_{y^{-1}g}$. If $T \not\leq P$ then $T \not\leq P^y$, and then (2) applies to $(y^{-1}g, P^y)$ in the role of (g, P) and yields a contradiction. So, (3) holds.

With these preliminaries in place, we now assume that g is a counter-example (i.e. $T \not\leq P$) with $\dim(P)$ is as large as possible. By 3.5 and 3.10 there exists $f \in \mathcal{L}$ such that $Q^f \leq S$ and such that $N_S(Q^f) \in \text{Syl}_p(N_{\mathcal{L}}(Q^f))$. Set $h = gf$ (where the product is defined via P) and set $R = P^h$. Let (h', P') be maximal in $\mathcal{L} \circ \Delta$ with $(h, P) \uparrow (h', P')$, and set $R' = R^{h'}$. If $T \leq P'$ then 4.7 yields $h = yh'$ for some $y \in \mathcal{N}$ such that $P \leq S_y$ and such that $N_T(P^y) \in \text{Syl}_p(N_{\mathcal{N}}(P^y))$. The existence of such an element y contradicts (3), so we conclude that $T \not\leq P'$. Then (h', P') is a counter-example to the proposition, and the maximality of $\dim(P)$ yields $\dim(P) = \dim(P')$. Then (h, P) is maximal in $\mathcal{L} \circ \Delta$, as is (h^{-1}, R) by 4.6(b), and so $T \not\leq R$. But $N_T(R) \in \text{Syl}_p(N_{\mathcal{N}}(R))$ since $R = Q^f$, and so (2) applies with (h^{-1}, R) in the role of (g, P) . Thus $T \leq R$, so $T \leq P$, and the proof is complete. \square

Lemma 4.9. *Suppose that $S = C_S(T)T$ and that $N_{\mathcal{N}}(T) \leq N_{\mathcal{L}}(S)$. Then every element of $N_{\mathcal{L}}(T)$ is \uparrow -maximal with respect to \mathcal{N} .*

Proof. Let $f \in N_{\mathcal{L}}(T)$ and set $P = S_f$. Then $T \leq P$ and $T \leq P^f$. Let $(g, Q) \in \mathcal{L} \circ \Delta$ with $(f, P) \uparrow (g, Q)$, and let $x, y \in \mathcal{N}$ be chosen as in definition 4.5. Then $P \leq Q$ by 4.8, and we have $P = P^x$ and $(P^f)^y = P^f \leq Q^g$ by 4.1(b). In order to show that f is \uparrow -maximal it suffices now to show that $P = Q$, and hence it suffices to show that $N_Q(P) \leq P$.

Set $D = N_Q(P)$. Then $D = C_D(T)T$ by the hypothesis on S , and then $[C_D(T), x]$ may be computed in the group $N_{\mathcal{L}}(P)$. By assumption, $x \in N_{\mathcal{L}}(S)$, so $[C_D(T), x] \leq C_S(T) \cap \mathcal{N} = Z(T)$, and hence $x \in N_{\mathcal{L}}(D)$. Similarly $y \in N_{\mathcal{L}}(D^g)$, and so $(x, g, y^{-1}) \in \mathbf{D}$ via D . As $xg = fy$ by the setup of definition 4.5, we have $xgy^{-1} = f$ by 1.3(g), and thus $D \leq S_f$. That is, $N_Q(P) \leq P$ as required. \square

If X and Y are subsets of \mathcal{L} then one has the notion of the product XY , introduced in section 1, as the set of all $\Pi(x, y)$ with $(x, y) \in \mathbf{D} \cap (X \times Y)$.

Corollary 4.10 (Frattini Lemma). *Let (\mathcal{L}, Δ, S) be a locality, let $\mathcal{N} \trianglelefteq \mathcal{L}$ be a partial normal subgroup, and let Λ be the set of \uparrow -maximal elements of \mathcal{L} with respect to \mathcal{N} . Then $\mathcal{L} = \mathcal{N}\Lambda = \Lambda\mathcal{N}$. In particular, we have $\mathcal{L} = N_{\mathcal{L}}(T)\mathcal{N} = \mathcal{N}N_{\mathcal{L}}(T)$.*

Proof. Let $f \in \mathcal{L}$, set $P = S_f$, and choose $(g, Q) \in \mathcal{L} \circ \Delta$ so that $(f, P) \uparrow (g, Q)$ and so that g is \uparrow -maximal. Then 4.6 yields $f = xg$ for some $x \in \mathcal{N}$, and then 4.2 shows that $f = gy$ where $y = x^g$. \square

The following result is fundamental to the theory being developed here. It was discovered and proved by Bernd Stellmacher, in his reading of an early draft of [Ch1]. The proof given here is essentially his.

Lemma 4.11 (Splitting Lemma). *Let $(x, f) \in \mathbf{D}$ with $x \in \mathcal{N}$ and with f \uparrow -maximal. Then $S_{(x, f)} = S_{xf} = S_{(f, xf)}$.*

Proof. Appealing to 4.2: Set $y = x^f$ and $g = xf$ (so that also $g = fy$), and set $Q = S_{(x, f)}$ (so that also $Q = S_{(f, y)}$). Thus $Q \leq S_f \cap S_g$. Also, 4.2(a) yields $Q = S_f \cap S_x$. Set

$$P_0 = N_{S_f}(Q), \quad P_1 = N_{S_g}(Q), \quad P = \langle P_0, P_1 \rangle,$$

and set $R = P_0 \cap P_1$. Then $Q \leq R$. In fact, 4.3(b) shows that $y = f^{-1}g$ and that $(R^f)^y = R^g$, so $R \leq Q$, and thus $P_0 \cap P_1 = Q$. Assume now that (x, f) is a counter-example to the lemma. That is, assume $Q < S_g$ (proper inclusion). Then $Q < P_1$ by (L2), and so $P_1 < P_0$. Thus:

$$(1) \quad P_1 < S_f.$$

Among all counter-examples, take (x, f) so that $\dim(Q)$ is as large as possible. We consider two cases, as follows.

CASE 1: $x \in N_{\mathcal{L}}(T)$.

As $f \in N_{\mathcal{L}}(T)$ by 4.8, we have $T \leq Q$, and then $x \in N_{\mathcal{L}}(Q)$ by 4.1(b). Thus $Q^g = Q^{xf} = Q^f$. Set $Q' = Q^g$. Then 2.2(b) yields an isomorphism $c_f : N_{\mathcal{L}}(Q) \rightarrow N_{\mathcal{L}}(Q')$. Here $f = x^{-1}g$ so $c_f = c_{x^{-1}} \circ c_g$ by 2.2(c). As $x \in N_{\mathcal{N}}(Q) \trianglelefteq N_{\mathcal{L}}(Q)$, we obtain $(P_1)^{x^{-1}} \leq N_{\mathcal{N}}(Q)P_1$, and then

$$(P_1)^f = ((P_1)^{x^{-1}})^g \leq (N_{\mathcal{N}}(Q)P_1)^g \leq N_{\mathcal{N}}(Q')N_S(Q').$$

Also $(P_0)^f \leq N_S(Q')$, so

$$(2) \quad P^f \leq N_{\mathcal{N}}(Q')N_S(Q').$$

Since $T \leq Q'$, T is a Sylow p -subgroup of $N_{\mathcal{N}}(Q')$ by 4.1(c), and thus $N_S(Q')$ is a Sylow p -subgroup of $N_{\mathcal{N}}(Q')N_S(Q')$. By (2) there is then an element $v \in N_{\mathcal{N}}(Q')$ such that $P^{fv} \leq N_S(Q')$. In particular, we have:

$$(3) \quad P_0 \leq S_{(f,v)} \text{ and } P \leq S_{fv}.$$

Set $u = v^{f^{-1}}$. Then $(u, f) \in \mathbf{D}$, and we then have $uf = fv$ and $S_{(u,f)} = S_{(f,v)}$ by 4.2. If $S_{(f,v)} = S_{fv}$ then $(f, v) \in \mathbf{D}$ via P , so that $P \leq S_f$, contrary to (1). Thus $S_{(f,v)} \neq S_{fv}$, and so (u, f) is a counter-example to the lemma. Then (3) and the maximality of $\dim(Q)$ in the choice of (x, f) yields $Q = P_0 = N_{S_f}(Q)$, and so $Q = S_f$. As $(f, Q) \uparrow (g, P)$ via $(x, \mathbf{1})$, we have a contradiction to the \uparrow -maximality of f .

CASE 2: The case $x \notin N_{\mathcal{L}}(T)$.

Let h be \uparrow -maximal, with $(g, S_g) \uparrow (h, S_h)$. Then $T \leq S_h$ by 4.9, and there exists $r \in \mathcal{N}$ with $g = rh$ by 4.8.

Set $w = (f^{-1}, x^{-1}, r, h)$, observe that $w \in \mathbf{D}$ via Q^g , and find

$$\Pi(w) = (f^{-1}x^{-1})(rh) = g^{-1}g = \mathbf{1}.$$

Then 2.3 yields $h = r^{-1}xf$. Since both f and h are in $N_{\mathcal{L}}(T)$, 2.2(c) yields $r^{-1}x \in N_{\mathcal{L}}(T)$, and so $r^{-1}x \in N_{\mathcal{N}}(T)$. Then Case 1 applies to $(r^{-1}x, f)$, and thus $S_h = S_{(r^{-1}x, f)} \leq S_f$ (using 4.2). By definition of \uparrow there exist $a, b \in \mathcal{N}$ such that one has the usual sort of commutative diagram:

$$\begin{array}{ccc} S_h & \xrightarrow{h} & S_{h^{-1}} \\ a \uparrow & & \uparrow b \\ S_g & \xrightarrow[g]{} & S_{g^{-1}} \end{array}$$

As $T \leq S_h$, 4.1(b) yields

$$S_g \leq S_g T = (S_g)^a T \leq S_h,$$

and so $S_g \leq S_f$. This again contradicts (1), and completes the proof. \square

The splitting lemma yields a useful criterion for partial normality, as follows.

Corollary 4.12. *Let \mathcal{L} be a locality, let $\mathcal{N} \trianglelefteq \mathcal{L}$, and let $\mathcal{K} \trianglelefteq \mathcal{N}$ be a partial normal subgroup of \mathcal{N} . Suppose that \mathcal{K} is $N_{\mathcal{L}}(T)$ -invariant. (I.e. $x^h \in \mathcal{K}$ for all $(h^{-1}, x, h) \in \mathbf{D}$ such that $x \in \mathcal{K}$ and $h \in N_{\mathcal{L}}(T)$.) Then $\mathcal{K} \trianglelefteq \mathcal{L}$.*

Proof. Let $x \in \mathcal{K}$ and let $f \in \mathcal{L}$ such that x^f is defined. By the Frattini Lemma, we may write $f = yg$ with $y \in \mathcal{N}$ and with g \uparrow -maximal, and then the splitting lemma yields $S_f = S_{(y,g)}$. Set $u = (f^{-1}, x, f)$ and $v = (g^{-1}, y^{-1}, x, y, g)$. Then $S_u = S_v \in \Delta$, and $x^f = \Pi(u) = \Pi(v) = (x^y)^g$. Thus $x^f \in \mathcal{K}$, and $\mathcal{K} \trianglelefteq \mathcal{L}$. \square

A subset X of \mathcal{L} of the form $\mathcal{N}f$, $f \in \mathcal{L}$, will be called a *coset* of \mathcal{N} . A coset $\mathcal{N}f$ is *maximal* if it is not a proper subset of any coset of \mathcal{N} .

Proposition 4.13. *Let \mathcal{N} be a partial normal subgroup of the locality \mathcal{L} .*

- (a) $\mathcal{N}f = f\mathcal{N}$ for all $f \in N_{\mathcal{L}}(T)$, and if f is \uparrow -maximal with respect to \mathcal{N} then $\mathcal{N}f = \mathcal{N}f\mathcal{N} = f\mathcal{N}$.
- (b) Let $f, g \in \mathcal{L}$. Then

$$(g, S_g) \uparrow (f, S_f) \iff \mathcal{N}g \subseteq \mathcal{N}f \iff g \in \mathcal{N}f.$$

- (c) $g \in \mathcal{L}$ is \uparrow -maximal relative to \mathcal{N} if and only if $\mathcal{N}g$ is a maximal coset of \mathcal{N} .
- (d) \mathcal{L} is partitioned by the set \mathcal{L}/\mathcal{N} of maximal cosets of \mathcal{N} .
- (e) Let $u := (g_1, \dots, g_n) \in \mathbf{D}$ and let $v := (f_1, \dots, f_n)$ be a sequence of \uparrow -maximal elements of \mathcal{L} such that $g_i \in \mathcal{N}f_i$ for all i . Then $v \in \mathbf{D}$, $TS_u \leq S_v$, and $\mathcal{N}\Pi(u) \subseteq \mathcal{N}\Pi(v)$.

Proof. (a): That $\mathcal{N}f = f\mathcal{N}$ for $f \in N_{\mathcal{L}}(T)$ is given by 4.2. Now let f be \uparrow -maximal relative to \mathcal{N} . Then $f \in N_{\mathcal{L}}(T)$ by 4.8. Let $x, y \in \mathcal{N}$ such that $(x, f, y) \in \mathbf{D}$. Then $(x, fy) \in \mathbf{D}$, and $(x, fy) = (x, y'f)$ where $y' = f y f^{-1}$. The splitting lemma (4.11) then yields $(x, y', f) \in \mathbf{D}$, and thus $\mathcal{N}f\mathcal{N} \subseteq \mathcal{N}f$. The reverse inclusion is obvious, and yields (a).

(b): Let $f, g \in \mathcal{L}$ and suppose that $(g, S_g) \uparrow (f, S_f)$. Then $g = xf$ for some $x \in \mathcal{N}$ by 4.10, and then 4.11 yields $S_g = S_{(x,f)} \leq S_f$. Let $y \in \mathcal{N}$ such that $(y, g) \in \mathbf{D}$. Since $(S_{(y,g)})^y \leq S_g = S_{(x,f)}$ we get $(y, x, f) \in \mathbf{D}$, and $yg = (yx)f \in \mathcal{N}f$. Thus $(g, S_g) \uparrow (f, S_f) \implies \mathcal{N}g \subseteq \mathcal{N}f$. Clearly $\mathcal{N}g \subseteq \mathcal{N}f \implies g \in \mathcal{N}f$. The required circle of implications is then completed by 4.11.

(c): Immediate from (b).

(d): Let f and g be \uparrow -maximal, and let $h \in \mathcal{N}f \cap \mathcal{N}g$. Thus there exist $x, y \in \mathcal{N}$ with $(x, f) \in \mathbf{D}$, $(y, g) \in \mathbf{D}$, and with $h = xf = yg$. Then also $(x^{-1}, x, f) \in \mathbf{D}$, so $(x^{-1}, yg) = (x^{-1}, xf) \in \mathbf{D}$. The splitting lemma then yields $(x^{-1}, y, g) \in \mathbf{D}$, and we thereby obtain $f = x^{-1}yg \in \mathcal{N}g$. Now (b) implies that $\mathcal{N}f \subseteq \mathcal{N}g$, and symmetry gives the reverse inclusion. Thus $\mathcal{N}f = \mathcal{N}g$ if $\mathcal{N}f \cap \mathcal{N}g \neq \emptyset$.

(e): Let $x_i \in \mathcal{N}$ with $g_i = x_i f_i$. Set $Q_0 = S_u$, and set $Q_i = (Q_{i-1})^{g_i}$ for $1 \leq i \leq n$. Then $Q_{i-1} \leq S_{g_i} = S_{(x_i, f_i)} \leq S_{f_i}$ by 4.11, and then also $Q_{i-1}T = (Q_{i-1})^{x_i}T \leq S_{f_i}$, by 4.1(b)

and 4.8. Thus:

$$(Q_{i-1}T)^{f_i} = (Q_{i-1})^{x, f_i}T = Q_iT \leq S_{f_{i+1}},$$

and $v \in \mathbf{D}$ via Q_0T . Now $\Pi(u) \in \mathcal{N}\Pi(v)$ by the Frattini calculus (4.4), and then (b) implies that $\mathcal{N}Pi(u) \subseteq \mathcal{N}\Pi(v)$. \square

Let \equiv be the equivalence relation on \mathcal{L} defined by the partition in 4.13(d). In view of points (a) and (b) of 4.13 we refer to the \equiv -classes as the *maximal cosets* of \mathcal{N} in \mathcal{L} .

Lemma 4.14. *Let \mathcal{H} be a partial subgroup of the locality \mathcal{L} , containing the partial normal subgroup $\mathcal{N} \trianglelefteq \mathcal{L}$. Then \mathcal{H} is the disjoint union of the maximal cosets of \mathcal{N} contained in \mathcal{H} .*

Proof. Let $f \in \mathcal{H}$. Apply the Frattini lemma (4.10) to obtain $f = xh$ for some $x \in \mathcal{N}$ and some $h \in N_{\mathcal{L}}(T)$ such that h is \uparrow -maximal with respect to \mathcal{N} . Then $h = x^{-1}f$ by 1.3(d), and thus $h \in \mathcal{H}$ as $\mathcal{N} \leq \mathcal{H}$. Then also $\mathcal{N}h \subseteq \mathcal{H}$, where $\mathcal{N}h$ is a maximal coset of \mathcal{N} by 4.13(b). \square

The set \mathcal{L}/\mathcal{N} of maximal cosets of \mathcal{N} may also be denoted $\overline{\mathcal{L}}$. Let $\rho : \mathcal{L} \rightarrow \overline{\mathcal{L}}$ be the mapping which sends $g \in \mathcal{L}$ to the unique maximal coset of \mathcal{N} containing g . Set $\mathbf{W} := \mathbf{W}(\mathcal{L})$ and $\overline{\mathbf{W}} = \mathbf{W}(\overline{\mathcal{L}})$, and let $\rho^* : \mathbf{W} \rightarrow \overline{\mathbf{W}}$ be the induced mapping of free monoids. For any subset or element X of \mathbf{W} , write \overline{X} for the image of X under ρ^* , and similarly if Y is a subset or element of \mathcal{L} write \overline{Y} for the image of Y under ρ . In particular, $\overline{\mathbf{D}}$ is the image of \mathbf{D} under ρ^* .

For $w \in \mathbf{W}$, we shall say that w is \uparrow -maximal if every entry of w is \uparrow -maximal.

Lemma 4.15. *There is a unique mapping $\overline{\Pi} : \overline{\mathbf{D}} \rightarrow \overline{\mathcal{L}}$, a unique involutory bijection $\overline{g} \mapsto \overline{g}^{-1}$ on $\overline{\mathcal{L}}$, and a unique element $\overline{\mathbf{I}}$ of $\overline{\mathcal{L}}$ such that $\overline{\mathcal{L}}$, with these structures, is a partial group, and such that ρ is a homomorphism of partial groups. Moreover, we have $\text{Ker}(\rho) = \mathcal{N}$, and the homomorphism of free monoids $\rho^* : \mathbf{W}(\mathcal{L}) \rightarrow \mathbf{W}(\overline{\mathcal{L}})$ maps $\mathbf{D}(\mathcal{L})$ onto $\mathbf{D}(\overline{\mathcal{L}})$.*

Proof. Let $u = (g_1, \dots, g_n)$ and $v = (h_1, \dots, h_n)$ be members of \mathbf{D} such that $\overline{u} = \overline{v}$. By 4.13(d) there exists, for each i , an \uparrow -maximal $f_i \in \mathcal{L}$ with $g_i, h_i \in \mathcal{N}f_i$. Set $w = (f_1, \dots, f_n)$. Then $w \in \mathbf{D}$ by 4.13(e), and then 4.3(a) shows that $\Pi(u)$ and $\Pi(v)$ are elements of $\mathcal{N}\Pi(w)$. Thus $\overline{\Pi(u)} = \overline{\Pi(w)} = \overline{\Pi(v)}$, and there is a well-defined mapping $\overline{\Pi} : \overline{\mathbf{D}} \rightarrow \overline{\mathcal{L}}$ given by

$$(*) \quad \overline{\Pi}(w) = \overline{\Pi(w)}.$$

For any subset X of \mathcal{L} write X^{-1} for the set of inverses of elements of X . For any $f \in \mathcal{L}$ we then have $(\mathcal{N}f)^{-1} = f^{-1}\mathcal{N}^{-1}$ by 1.1(4). Here $\mathcal{N}^{-1} = \mathcal{N}$ as \mathcal{N} is a partial group, and then $(\mathcal{N}f)^{-1} = \mathcal{N}f^{-1}$ by 4.13(a). The inversion map $\mathcal{N}f \mapsto \mathcal{N}f^{-1}$ is then well-defined, and is an involutory bijection on $\overline{\mathcal{L}}$. Set $\overline{\mathbf{I}} = \mathcal{N}$.

We now check that the axioms in 1.1, for a partial group, are satisfied by the above structures. Since $\overline{\mathbf{D}}$ is the image of \mathbf{D} under ρ^* , we get $\overline{\mathcal{L}} \subseteq \overline{\mathbf{D}}$. Now let $\overline{w} = \overline{u} \circ \overline{v} \in \overline{\mathbf{D}}$, let u, v be \uparrow -maximal pre-images in \mathbf{W} of \overline{u} , and \overline{v} , and set $w = u \circ v$. Then w is \uparrow -maximal,

and so $w \in \mathbf{D}$ by 4.13(e). Then u and v are in \mathbf{D} , and so \bar{u} and \bar{v} are in $\bar{\mathbf{D}}$. Thus $\bar{\mathbf{D}}$ satisfies 1.1(1). Clearly, (*) implies that $\bar{\Pi}$ restricts to the identity on $\bar{\mathcal{L}}$, so $\bar{\Pi}$ satisfies 1.1(2).

Next, let $\bar{u} \circ \bar{v} \circ \bar{w} \in \bar{\mathbf{D}}$, and choose corresponding \uparrow -maximal pre-images u, v, w . Set $g = \Pi(v)$. Then $\bar{g} = \bar{\Pi}(\bar{v})$ by (*). By 1.1(3) we have both $u \circ v \circ w$ and $u \circ (g) \circ w$ in \mathbf{D} , and these two words have the same image under Π . Applying ρ^* we obtain words in $\bar{\mathbf{D}}$ having the same image under $\bar{\Pi}$, and thus $\bar{\Pi}$ satisfies 1.1(3). By definition, $\bar{\Pi}(\emptyset) = \bar{\mathbf{I}}$, and then the condition 1.1(4) is readily verified. Thus, $\bar{\mathcal{L}}$ is a partial group.

By definition, $\bar{\mathbf{D}}$ is the image of \mathbf{D} under ρ^* . So, in order to check that ρ is a homomorphism of partial groups it suffices to show that if $w \in \mathbf{D}$ then $\bar{\Pi}(w\rho^*) = \Pi(w)\rho$. But this is simply the statement (*). Moreover, it is this observation which establishes that the given partial group structure on $\bar{\mathcal{L}}$ is the unique one for which ρ is a homomorphism of partial groups. We have $f \in \text{Ker}(\rho)$ if and only if $f\gamma = \bar{\mathbf{I}} = \mathcal{N}$. Since $\mathcal{N}f \subseteq \mathcal{N}$ implies $f \in \mathcal{N}$, and since \mathcal{N} is the maximal coset of \mathcal{L} containing $\mathbf{1}$, we obtain $\text{Ker}(\rho) = \mathcal{N}$. \square

Section 5: Quotient localities

We continue the setup of the preceding section, where (\mathcal{L}, Δ, S) is a fixed locality and $\mathcal{N} \trianglelefteq \mathcal{L}$ is a partial normal subgroup. We have seen in 4.15 that the set \mathcal{L}/\mathcal{N} of maximal cosets of \mathcal{N} inherits from \mathcal{L} a partial group structure via the projection map $\rho : \mathcal{L} \rightarrow \mathcal{L}/\mathcal{N}$. The aim now is to go further, and to show that \mathcal{L}/\mathcal{N} is a locality.

Lemma 5.1. *Let (\mathcal{L}, Δ, S) be a locality and let $\mathcal{N} \trianglelefteq \mathcal{L}$. Then $(\mathcal{N}S, \Delta, S)$ is a locality.*

Proof. It follows from 2.17 that $\mathcal{N}S$ is a partial subgroup of \mathcal{L} . One observes that $\mathbf{D}(\mathcal{N}S)$ is the subset \mathbf{D}_Δ of $\mathbf{W}(\mathcal{N}S)$, as defined in 2.1, and this suffices to show that $(\mathcal{N}S, \Delta)$ is objective. Thus $(\mathcal{N}S, \Delta, S)$ is a pre-locality, and is finite-dimensional by 2.16(b). The conditions (L1) and (L2) in definition 3.1 are inherited by $\mathcal{N}S$ from \mathcal{L} in an obvious way, so $(\mathcal{N}S, \Delta, S)$ is a locality. \square

Lemma 5.2. *Let $P \leq S$ be a subgroup of S , and let X be a subset of S containing P and having the property that $P^x \subseteq X$ for all $x \in X$. Then either $X = P$ or P is a proper subset of $X \cap N_S(P^*)$.*

Proof. Let $\{S_n\}$ be a framing of S by finite subgroups, and set $P_n = P \cap S_n$ and $X_n = X \cap S_n$. Then $(P_n)^x \subseteq X_n$ for all $x \in X_n$. By finite dimensionality, $P^* = (P_n)^*$ for all sufficiently large n , so we may start the indexing so that $P^* = (P_0)^*$. Notice that $X_n \subseteq X_{n+1}$ for all n and that $X = \bigcup \{X_n\}$. Assuming that $P \neq X$, there is then an index n such that $P_n \neq X_n$. Again, we may assume that $n = 0$.

Let \mathcal{C} be the set of subgroups Q of S_0 such that $X_0 \cap Q = P_0$. Then $P_0 \in \mathcal{C}$, and since S_0 is finite there exists a maximal $Q \in \mathcal{C}$ with respect to inclusion. As $P_0 \neq X_0$ we have $Q \neq S_0$, and so $Q < N_{S_0}(Q)$. Thus $N_{S_0}(Q) \notin \mathcal{C}$, and so there exists $x \in X_0 \cap N_{S_0}(Q)$ with $x \notin P_0$. Then $(P_0)^x \subseteq X_0 \cap Q = P_0$, and so $(P_0)^x = P_0$. Then $x \in N_S(P^*)$ by 2.13, and $x \notin P$ since $P \cap X_0 \subseteq P \cap S_0 = P_0$. \square

Theorem 5.3. *Let (\mathcal{L}, Δ, S) be a locality, let $\overline{\mathcal{L}}$ be a partial group, and let $\beta : \mathcal{L} \rightarrow \overline{\mathcal{L}}$ be a homomorphism of partial groups such that the induced map $\beta^* : \mathbf{W}(\mathcal{L}) \rightarrow \mathbf{W}(\overline{\mathcal{L}})$ of free monoids restricts to a surjection $\mathbf{D}(\mathcal{L}) \rightarrow \mathbf{D}(\overline{\mathcal{L}})$. Set $\mathcal{N} = \text{Ker}(\beta)$ and $T = S \cap \mathcal{N}$. Further, set $\mathbf{D} = \mathbf{D}(\mathcal{L})$, $\overline{\mathbf{D}} = \mathbf{D}(\overline{\mathcal{L}})$, $\overline{S} = S\beta$, and $\overline{\Delta} = \{P\beta \mid P \in \Delta\}$. Then $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is a locality. Moreover:*

- (a) *The fibers of β are the maximal cosets of \mathcal{N} .*
- (b) *For each $\overline{w} \in \mathbf{W}(\overline{\mathcal{L}})$ there exists $w \in \mathbf{W}(\mathcal{L})$ such that $\overline{w} = w\beta^*$ and such that each entry of w is \uparrow -maximal relative to the partial normal subgroup \mathcal{N} of \mathcal{L} . For any such w we then have $\overline{S_w} = \overline{S_{\overline{w}}}$, and $w \in \mathbf{D}$ if and only if $\overline{w} \in \overline{\mathbf{D}}$.*
- (c) *Let $P, Q \in \Delta$ with $T \leq P \cap Q$. Then β restricts to a surjection $N_{\mathcal{L}}(P, Q) \rightarrow N_{\overline{\mathcal{L}}}(P\beta, Q\beta)$; and to a surjective homomorphism if $P = Q$.*
- (d) *β is an isomorphism if and only if $\mathcal{N} = 1$.*

Proof. The hypothesis that $\mathbf{D}\beta^* = \overline{\mathbf{D}}$ implies that β^* maps the set of words of length 1 in \mathcal{L} onto the set of words of length 1 in $\overline{\mathcal{L}}$. Thus β is surjective.

Let M be a subgroup of \mathcal{L} . The restriction of β to M is then a homomorphism of partial groups, and hence a homomorphism $M \rightarrow M\beta$ of groups by 1.13. In particular, \overline{S} is a p -group, and $\overline{\Delta}$ is a set of subgroups of \overline{S} .

We have $\mathcal{N} \leq \mathcal{L}$ by 1.14. Let Λ be the set of elements $g \in \mathcal{L}$ such that g is \uparrow -maximal relative to \mathcal{N} . For any $g \in \Lambda$, β is constant on the maximal coset $\mathcal{N}g$ (see 4.14) of \mathcal{N} , so β restricts to a surjection of Λ onto $\overline{\mathcal{L}}$. This shows that β^* restricts to a surjection of $\mathbf{W}(\Lambda)$ onto $\mathbf{W}(\overline{\mathcal{L}})$. If $\overline{w} \in \overline{\mathbf{D}}$ then there exists $w \in \mathbf{D}$ with $w\beta^* = \overline{w}$, and then 4.13(e) shows that such a w may be chosen to be in $\mathbf{W}(\Lambda)$. Set $\mathbf{D}(\Lambda) = \mathbf{D} \cap \mathbf{W}(\Lambda)$. Thus:

- (1) β^* maps $\mathbf{D}(\Lambda)$ onto $\overline{\mathbf{D}}$.

Let $\overline{w} \in \overline{\mathbf{D}}$, let $w \in \mathbf{D}(\Lambda)$ with $w\beta^* = \overline{w}$, let $\overline{a}, \overline{b} \in \overline{S}$, and let $a, b \in S$ with $a\beta = \overline{a}$ and $b\beta = \overline{b}$. Then $a, b \in \Lambda$ by 4.6(a), and $(a) \circ w \circ (b) \in \mathbf{D}(\Lambda)$ by 2.17. Then $(\overline{a}) \circ \overline{w} \circ (\overline{b}) \in \overline{\mathbf{D}}$. This shows:

- (2) $\overline{\mathbf{D}}$ is a \overline{S} -biset (cf. 2.7).

Let $\overline{a} \in \overline{S}$, let $a \in S$ be a preimage of \overline{a} , and let $h \in \mathcal{L}$ be any preimage of \overline{a} . Then $(a, h^{-1}) \in \mathbf{D}$, and $ah^{-1} \in \mathcal{N}$, so:

- (3) The β -preimage of an element $\overline{a} \in \overline{S}$ is a maximal coset $\mathcal{N}a$, where $a \in S$.

Fix $\overline{g} \in \overline{\mathcal{L}}$, let $g \in \Lambda$ with $g\beta = \overline{g}$, set $P = S_g$, and set

$$\overline{S_{\overline{g}}} = \{\overline{x} \in \overline{S} \mid \overline{x}^{\overline{S}} \in \overline{S}\}.$$

Let $\overline{a} \in \overline{S_{\overline{g}}}$ and set $\overline{b} = \overline{a}^{\overline{g}}$. As in the proof of 2.7, we show that $\overline{P}^{\overline{a}} \subseteq \overline{S_{\overline{g}}}$. Namely, from $(\overline{g}^{-1}, \overline{a}, \overline{g}) \in \overline{\mathbf{D}}$ and $\overline{P}i(\overline{g}^{-1}, \overline{a}, \overline{g}) = \overline{b}$ we obtain (from two applications of (2)):

$$(*) \quad (\overline{P}^{\overline{a}})^{\overline{g}} = \overline{P}^{\overline{b}} \leq \overline{S}.$$

Thus, the set $\overline{\Gamma}$ of all $\overline{S_{\overline{g}}}$ -conjugates of \overline{P} is a set of subgroups of $\overline{S_{\overline{g}}}$. Let Γ be the set of all preimages in S of members of $\overline{\Gamma}$, and set $X = \bigcup \Gamma$. Then $X = X^{-1}$ is a subset of S ,

containing P , and having the property that $P^x \subseteq X$ for all $x \in X$. As $P = S_g \in \Omega$, 5.2 implies that either $X = P$ or P is properly contained in $N_S(P) \cap S$.

Assuming now that $\overline{P} \neq \overline{S}_{\overline{g}}$, we may choose an element $x \in N_X(P) - P$. Set $Q = P\langle x \rangle$. Then Q^g is defined (and is a subgroup of $N_{\mathcal{L}}(P^g)$) by 2.3(b). As $\overline{Q}^{\overline{g}} \leq \overline{S}$ we obtain $Q^g \leq \mathcal{N}S$. As $\mathcal{N}S$ is a locality by 5.1, S is a Sylow p -subgroup of $\mathcal{N}S$ by 3.8. Thus there exists $f \in \mathcal{N}$ with $(Q^g)^f \leq S$. Here $(g, f) \in \mathbf{D}$ via P , so $Q \leq S_{gf}$. This contradicts the \uparrow -maximality of g , so we conclude:

$$(4) \quad (S_g)\beta = \overline{S}_{g\beta} \text{ for each } g \in \Lambda.$$

For $\overline{w} \in \mathbf{W}(\overline{\mathcal{L}})$ define $\overline{S}_{\overline{w}}$ to be the set of all $\overline{x} \in \overline{S}$ such that \overline{x} is conjugated successively into \overline{S} by the entries of \overline{w} . As an immediate consequence of (4):

$$(5) \quad (S_w)\beta = \overline{S}_{w\beta^*} \text{ for all } w \in \mathbf{W}(\Lambda).$$

Notice that (5) implies point (b).

We now verify that $(\overline{\mathcal{L}}, \overline{\Delta})$ is objective. Thus, let $\overline{w} \in \mathbf{W}(\overline{\mathcal{L}})$ with $\overline{S}_{\overline{w}} \in \overline{\Delta}$. Let $w \in \mathbf{W}(\Lambda)$ with $w\beta^* = w'$. Then (5) yields $S_w \in \Delta$, so $w \in \mathbf{D}$, and hence $\overline{w} \in \overline{\mathbf{D}}$. Thus $(\overline{\mathcal{L}}, \overline{\Delta})$ satisfies the condition (O1) in definition 2.1 of objectivity. Now let $\overline{P} \in \overline{\Delta}$, let $\overline{f} \in N_{\overline{\mathcal{L}}}(\overline{P}, \overline{S})$, and set $\overline{Q} = (\overline{P})^{\overline{f}}$. Then (4) yields $\overline{Q} = Q\beta$ for some $Q \in \Delta$, and thus $\overline{Q} \in \overline{\Delta}$. Any overgroup of \overline{Q} in \overline{S} is the image of an overgroup of Q in S as β maps S onto \overline{S} , so $(\overline{\mathcal{L}}, \overline{\Delta})$ satisfies (O2) in 2.1. Thus $(\overline{\mathcal{L}}, \overline{\Delta})$ is objective. As $\overline{\Delta}$ is a set of subgroups of \overline{S} , $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is then a pre-locality, and finite-dimensional since, by (5), the poset $\Omega_{\overline{S}}(\overline{\mathcal{L}})$ is a homomorphic image of $\Omega_S(\mathcal{L})$.

Let $P, Q \in \Delta$ with $T \leq P \cap Q$, set $\overline{P} = P\beta$ and $\overline{Q} = Q\beta$, and let $\overline{g} \in \overline{\mathcal{L}}$ such that $\overline{P}^{\overline{g}}$ is defined and is a subset of \overline{Q} . Let g be a preimage of \overline{g} in Λ . Then $P \leq S_g$ by (4). As β is a homomorphism, $(P^g)\beta$ is a subgroup of \overline{Q} . Then $P^g \leq Q$ since β restricts to an epimorphism $S \rightarrow \overline{S}$ with kernel T . As β maps subgroups of \mathcal{L} homomorphically to subgroups of $\overline{\mathcal{L}}$ (by 1.13) we obtain (c). In the case that $P = Q = S$ we obtain in this way an epimorphism from $N_{\mathcal{L}}(S)$ to $N_{\overline{\mathcal{L}}}(\overline{S})$. As S is a Sylow p -subgroup of $N_{\mathcal{L}}(S)$ it follows that \overline{S} is a maximal p -subgroup of $\overline{\mathcal{L}}$. Thus $\overline{\mathcal{L}}$ satisfies the condition (L1) in definition 3.1. The condition (L2) is inherited by $\overline{\mathcal{L}}$ in an obvious way, and hence $\overline{\mathcal{L}}$ is a locality.

Let $g, h \in \Lambda$ with $\gamma\beta = h\beta$. Then $S_g = S_h$ by (4), so $(g, h^{-1}) \in \mathbf{D}$, and $(gh^{-1})\beta = \mathbf{1}$. Thus $g \in \mathcal{N}h$, and then $\mathcal{N}g = \mathcal{N}h$ by 4.14(b). This yields (a), and it remains only to prove (d).

If β is an isomorphism then β is injective, and $\mathcal{N} = \mathbf{1}$. On the other hand, suppose that $\mathcal{N} = \mathbf{1}$. Then (a) shows that β is injective, and so β is a bijection. That β^{-1} is then a homomorphism of partial groups is given by (5). Thus (d) holds, and the proof is complete. \square

Definition 5.4. Let (\mathcal{L}, Δ, S) and $(\mathcal{L}', \Delta', S')$ be localities, and let $\beta : \mathcal{L} \rightarrow \mathcal{L}'$ be a homomorphism of partial groups. Then β is a *projection* if:

- (1) $\mathbf{D}\beta^* = \mathbf{D}'$, and
- (2) $\Delta' = \{P\beta \mid P \in \Delta\}$.

Corollary 5.5. *Let (\mathcal{L}, Δ, S) be a locality, let $\mathcal{N} \trianglelefteq \mathcal{L}$ be a partial normal subgroup and let $\rho : \mathcal{L} \rightarrow \mathcal{L}/\mathcal{N}$ be the mapping which sends $g \in \mathcal{L}$ to the unique maximal coset of \mathcal{N} containing g . Set $\overline{\mathcal{L}} = \mathcal{L}/\mathcal{N}$, set $\overline{S} = S\rho$, and let $\overline{\Delta}$ be the set of images under ρ of the members of Δ . Regard $\overline{\mathcal{L}}$ as a partial group in the unique way (given by 4.15) which makes ρ into a homomorphism of partial groups. Then $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is a locality, and ρ is a projection.*

Proof. Immediate from 4.15 and 5.3. \square

Theorem 5.6 (“First Isomorphism Theorem”). *Let (\mathcal{L}, Δ, S) and $(\mathcal{L}', \Delta', S')$ be localities, let $\beta : \mathcal{L} \rightarrow \mathcal{L}'$ be a projection, and let $\mathcal{N} \trianglelefteq \mathcal{L}$ be a partial normal subgroup of \mathcal{L} contained in $\text{Ker}(\beta)$. Let $\rho : \mathcal{L} \rightarrow \mathcal{L}/\mathcal{N}$ be the projection given by 5.3. Then there exists a unique homomorphism*

$$\gamma : \mathcal{L}/\mathcal{N} \rightarrow \mathcal{L}'$$

such that $\rho \circ \gamma = \beta$; and γ is then a projection. Moreover, γ is an isomorphism if and only if $\mathcal{N} = \text{Ker}(\beta)$.

Proof. Set $\mathcal{M} = \text{Ker}(\beta)$, and let $h \in \mathcal{L}$ be \uparrow -maximal relative to \mathcal{M} . Then $\mathcal{M}h$ is a maximal coset of \mathcal{M} in \mathcal{L} by 4.13(b), and $\mathcal{M}h = \mathcal{M}h\mathcal{M}$ by 4.13(a). Let $g \in \mathcal{M}h$. As $\mathcal{N} \leq \mathcal{M}$ by hypothesis, the splitting lemma (4.11, as applied to \mathcal{M} and h) yields

$$\mathcal{N}g\mathcal{N} \subseteq \mathcal{N}(\mathcal{M}h)\mathcal{N} \subseteq \mathcal{M}(\mathcal{M}h)\mathcal{M} = \mathcal{M}h\mathcal{M}.$$

The definition 4.6 of the relation \uparrow on $\mathcal{L} \circ \Delta$ shows that $\mathcal{N}g\mathcal{N}$ contains an element f which is \uparrow -maximal with respect to \mathcal{N} . Then $f \in \mathcal{M}h\mathcal{M}$, so $f \in \mathcal{M}h$, and another application of the splitting lemma yields $\mathcal{N}f \subseteq \mathcal{M}h$. Here $\mathcal{N}f$ is the maximal coset of \mathcal{N} containing f . We have thus shown:

(*) The partition \mathcal{L}/\mathcal{N} of \mathcal{L} is a refinement of the partition \mathcal{L}/\mathcal{M} .

By 5.3(a) β induces a bijection $\mathcal{L}/\mathcal{M} \rightarrow \mathcal{L}'$. Set $\overline{\mathcal{L}} = \mathcal{L}/\mathcal{N}$. Then (*) implies that there is a mapping $\gamma : \overline{\mathcal{L}} \rightarrow \mathcal{L}'$ which sends the maximal coset $\mathcal{N}f\mathcal{N}$ to $f\beta$. Clearly, γ is the unique mapping $\overline{\mathcal{L}} \rightarrow \mathcal{L}'$ such that $\rho \circ \gamma = \beta$.

Let $\overline{w} \in \mathbf{D}(\overline{\mathcal{L}})$. Then 5.3(b) yields a word $w \in \mathbf{D}$ such that $w\rho^* = \overline{w}$ and such that the entries of w are \uparrow -maximal relative to \mathcal{N} . We have $\overline{w}\gamma^* = w\beta^*$, so γ^* maps $\mathbf{D}(\overline{\mathcal{L}})$ into $\mathbf{D}(\mathcal{L}')$. Let Π' and $\overline{\Pi}$ be the products in \mathcal{L}' and $\overline{\mathcal{L}}$, respectively. As β and ρ are homomorphisms we get

$$\Pi'(\overline{w}\gamma^*) = \Pi'(w\beta^*) = (\Pi(w))\beta = (\overline{\Pi(w)})\gamma = (\overline{\Pi}(\overline{w}))\gamma,$$

and thus γ is a homomorphism. As $\beta = \rho \circ \gamma$ is a projection, and ρ is a projection, one verifies that γ^* maps $\mathbf{D}(\overline{\mathcal{L}})$ onto $\mathbf{D}(\mathcal{L}')$ and that γ maps $\overline{\Delta}$ onto Δ' . Thus γ is a projection.

We have $\mathcal{M} = \mathcal{N}$ if and only if $\text{Ker}(\gamma) = \mathbf{1}$. Then 5.3(d) shows that γ is an isomorphism if and only if $\mathcal{M} = \mathcal{N}$; completing the proof. \square

Proposition 5.7 (Partial Subgroup Correspondence). *Let (\mathcal{L}, Δ, S) and $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ be localities, and let $\beta : \mathcal{L} \rightarrow \overline{\mathcal{L}}$ be a projection. Set $\mathcal{N} = \text{Ker}(\beta)$ and set $T = S \cap \mathcal{N}$. Then β induces a bijection σ from the set \mathfrak{H} of partial subgroups \mathcal{H} of \mathcal{L} containing \mathcal{N} to the set $\overline{\mathfrak{H}}$ of partial subgroups $\overline{\mathcal{H}}$ of $\overline{\mathcal{L}}$. Moreover, for any $\mathcal{H} \in \mathfrak{H}$, we have $\mathcal{H}\beta \trianglelefteq \mathcal{L}'$ if and only if $\mathcal{H} \trianglelefteq \mathcal{L}$.*

Proof. Any partial subgroup of \mathcal{L} containing \mathcal{N} is a union of maximal cosets of \mathcal{N} by 4.14. Then 5.3(a) enables the same argument that one has for groups, for proving that ρ induces a bijection $\mathfrak{H} \rightarrow \overline{\mathfrak{H}}$. Since each maximal coset of \mathcal{N} contains an element which is \uparrow -maximal with respect to \mathcal{N} , one may apply 5.3(b) in order to show that a partial subgroup $\mathcal{H} \in \mathfrak{H}$ is normal in \mathcal{L} if and only if its image is normal in $\overline{\mathcal{L}}$. \square

5.8 Remark. A comprehensive “second isomorphism theorem” appears to be out of reach, for two reasons. First, given a partial subgroup $\mathcal{H} \leq \mathcal{L}$ and a partial normal subgroup $\mathcal{N} \trianglelefteq \mathcal{L}$, there appears to be no reason why the image of \mathcal{H} under the projection $\rho : \mathcal{L} \rightarrow \mathcal{L}/\mathcal{N}$ should be a partial subgroup of \mathcal{L}/\mathcal{N} , other than in special cases. Second, there seems to be no way, in general, to define the quotient of \mathcal{H} over the partial normal subgroup $\mathcal{H} \cap \mathcal{N}$ of \mathcal{H} . On the other hand, a “third isomorphism theorem” may easily be deduced from 4.4 and from the observation that a composition of projections is again a projection.

Lemma 5.9. *Let $\mathcal{N} \trianglelefteq \mathcal{L}$ and let $\rho : \mathcal{L} \rightarrow \mathcal{L}/\mathcal{N}$ be the canonical projection. Further, let \mathcal{H} be a partial subgroup of \mathcal{L} containing \mathcal{N} and let X be an arbitrary subset of \mathcal{L} . Then $(X \cap \mathcal{H})\rho = X\rho \cap \mathcal{H}\rho$.*

Proof. By 4.14, \mathcal{H} is a union of maximal cosets of \mathcal{N} , and then $\mathcal{H}\rho$ is the set of those maximal cosets. On the other hand $X\rho$ is the set of all maximal cosets $\mathcal{N}g$ of \mathcal{N} such that $X \cap \mathcal{N}g \neq \emptyset$. Thus $X\rho \cap \mathcal{H}\rho \subseteq (X \cap \mathcal{H})\rho$. The reverse inclusion is obvious. \square

Corollary 5.10. *Let $\mathcal{N} \trianglelefteq \mathcal{L}$, and let \mathcal{M} be a partial normal subgroup of \mathcal{L} containing \mathcal{N} . Let $\rho : \mathcal{L} \rightarrow \mathcal{L}/\mathcal{N}$ be the canonical projection. Then $(S \cap \mathcal{M})\rho$ is a maximal p -subgroup of $\mathcal{M}\rho$.*

Proof. Write $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ for the quotient locality given by 5.5, and set $\overline{\mathcal{M}} = \mathcal{M}\rho$. Applying 5.9 with S in the role of X , we obtain $(S \cap \mathcal{M})\rho = \overline{S} \cap \overline{\mathcal{M}}$. Since $\overline{\mathcal{M}} \trianglelefteq \overline{\mathcal{L}}$, it follows from 4.1(c) that $\overline{S} \cap \overline{\mathcal{M}}$ is maximal in the poset of p -subgroups of $\overline{\mathcal{M}}$, completing the proof. \square

Proposition 5.11. *Let $\mathcal{N} \trianglelefteq \mathcal{L}$, set $T = S \cap \mathcal{N}$, and set $\mathcal{L}_T = N_{\mathcal{L}}(T)$. Set $\overline{\mathcal{L}} = \mathcal{L}/\mathcal{N}$ and let $\rho : \mathcal{L} \rightarrow \overline{\mathcal{L}}$ be the canonical projection. Then the partial subgroup \mathcal{L}_T of \mathcal{L} is a locality $(\mathcal{L}_T, \Delta, S)$, and the restriction of ρ to \mathcal{L}_T is a projection $\mathcal{L}_T \rightarrow \overline{\mathcal{L}}$.*

Proof. That \mathcal{L}_T is a partial subgroup of \mathcal{L} having the structure of a locality $(\mathcal{L}_T, \Delta, S)$ is given by 3.13. Let ρ_T be the restriction of ρ to \mathcal{L}_T . Then ρ_T is a homomorphism of partial groups, and 4.14(e) shows that ρ_T maps $\mathbf{D}(\mathcal{L}_T)$ onto $\mathbf{D}(\overline{\mathcal{L}})$. That is, ρ_T is a projection. \square

Section 6: Products of partial normal subgroups

The main result of this section Theorem 6.7 is that the product of any collection of partial normal subgroups in a locality is again a partial normal subgroup. The proof, aside from some relatively minor details, is the same as that given by Ellen Henke [He] for finite localities. As in the finite case, the argument is based on the study of products of pairs of partial normal subgroups. For that reason, it will be convenient to establish the following notation.

Hypothesis 6.1. \mathcal{M} and \mathcal{N} are partial normal subgroups of the locality (\mathcal{L}, Δ, S) . Set $U = S \cap \mathcal{M}$ and $V = S \cap \mathcal{N}$. Also, set $\mathcal{K} = \mathcal{M} \cap \mathcal{N}$, and set $T = S \cap \mathcal{K}$. Let $\overline{\mathcal{L}}$ be the quotient locality \mathcal{L}/\mathcal{K} , and let $\rho : \mathcal{L} \rightarrow \overline{\mathcal{L}}$ be the canonical projection. Write \overline{X} for the image under ρ of a subset or element X of \mathcal{L} , and write $\overline{\mathbf{D}}$ for the domain of the product in $\overline{\mathcal{L}}$.

Lemma 6.2. *Assume the setup of 6.1, and suppose that $\mathcal{M} \cap \mathcal{N} \leq S$. Then $\mathcal{M} \leq N_{\mathcal{L}}(V)$ and $\mathcal{N} \leq N_{\mathcal{L}}(U)$.*

Proof. Let $g \in \mathcal{M}$, set $P = S_g$, and let $x \in N_V(P)$. Then $(x^{-1}, g^{-1}, x, g) \in \mathbf{D}$ via P^{gx} , and then

$$[x, g] \in \mathcal{M}\mathcal{N} \leq S \cap \mathcal{N} = V.$$

Thus $N_V(P) \leq P$, and then $V \leq P$ and $[V, g] \leq V$. That is, we have $\mathcal{M} \leq N_{\mathcal{L}}(V)$, and the lemma follows. \square

Lemma 6.3. *Assume the setup of 6.1, and suppose that $\mathcal{M} \cap \mathcal{N} = \mathbf{1}$.*

- (a) *For each $g \in \mathcal{M}\mathcal{N}$ there exists $x \in \mathcal{M}$ and $y \in \mathcal{N}$ such that $(x, y) \in \mathbf{D}$, $g = xy$, and $S_g = S_{(x, y)}$.*
- (b) *$\mathcal{M}\mathcal{N} = \mathcal{N}\mathcal{M}$ is a partial normal subgroup of \mathcal{L} , and $S \cap \mathcal{M}\mathcal{N} = UV$.*

Proof. Let \mathcal{T} be the set of all triples $(g, x, y) \in \mathcal{M}\mathcal{N} \times \mathcal{M} \times \mathcal{N}$ such that $g = xy$ and such that g is a counter-example to (a). Among all such triples, let (g, x, y) be chosen so that $\dim(S_{(x, y)})$ is as large as possible. Set $Q = S_{(x, y)}$ and set $P = N_{S_g}(Q)$. It suffices to show that $P = Q$ in order to obtain a contradiction and to thereby establish (a).

By 4.2 we have $(y, y^{-1}, x, y) \in \mathbf{D}$ and $g = yx^y$. Suppose that $P \leq S_y$. Then $P^y \leq S$, and since $P^g \leq S$ we conclude that $P \leq S_{(y, x^y)}$, and hence $P = Q$, as desired. Thus we may assume:

- (1) $P \not\leq S_y$.

Let h be \uparrow -maximal in the maximal coset of \mathcal{M} containing g . Then the Frattini Lemma (4.10) yields an element $r \in \mathcal{M}$ such that $g = rh$, and the Splitting Lemma (4.11) yields $S_g = S_{(r, h)}$. Then $Q \leq S_{(r, h)}$, so $(y^{-1}, x^{-1}, r, h) \in \mathbf{D}$ via Q^g and $\Pi(y^{-1}, x^{-1}, r, h) = \Pi(g^{-1}, g) = \mathbf{1}$. Thus:

$$(*) \quad y = x^{-1}rh \quad \text{and} \quad h = r^{-1}xy.$$

Since $y, h \in N_{\mathcal{L}}(U)$, it follows that $r^{-1}x \in N_{\mathcal{M}}(U)$, and then that $h = (r^{-1}x)y \in \mathcal{M}\mathcal{N}$.

Suppose that h does not provide a counter-example to (a). That is, suppose that there exists $x' \in \mathcal{M}$ and $y' \in \mathcal{N}$ such that $(x', y') \in \mathbf{D}$, $x'y' = h$, and $S_{(x', y')} = S_h$. As $r^{-1}xy = h = x'y'$ we get $xy = rx'y'$, and $(rx', y') \in \mathbf{D}$ with $rx'y' = rh = g$. The idea now is to replace (x, y) with (rx', y') and to contradict the assumption that $S_g \neq Q$. In order to achieve this, observe first of all that $S_g \leq S_r$ since $S_{(r, h)} = S_{rh} = S_g$. Then observe that $(S_g)^r \leq S_h$, and that $S_h = S_{(x', y')} \leq S_{x'}$. Thus $(S_g)^r \leq S_{x'}$, and so $S_g \leq S_{rx'}$. As $rx'y' = g$ we conclude that $S_g \leq S_{(rx', y')}$, which yields the desired contradiction. Thus h is itself a counter-example to (a).

Since $h = r^{-1}xy$ by (*), and since h and y are in $N_{\mathcal{L}}(U)$, we have $r^{-1}x \in N_{\mathcal{M}}(U)$, and then $U \leq S_{(r^{-1}x, y)}$ since $h \in N_{\mathcal{L}}(U)$. Note furthermore that $Q = S_{(x, y)} \leq S_g = S_{(r, h)} \leq S_r$, and thus $Q^r U \leq S_{(r^{-1}x, y)}$. The maximality of $\dim(Q)$ in our initial choice of (g, x, y) then yields $Q^r = Q^r U = S_{(r^{-1}x, y)}$. Thus $U^r \leq Q^r$, and conjugation by r^{-1} yields $U \leq Q$. A symmetric argument yields $V \leq Q$. Setting $H = N_{\mathcal{L}}(Q)$, it follows from 4.1(b) that $x, y \in H$. Then $Q = O_p(H)$.

Set $X = H \cap \mathcal{M}$ and $Y = H \cap \mathcal{N}$. Then X, Y , and UV are normal subgroups of H , and XY/UV is a p' -group. Set $\overline{H} = H/(X \cap Y)UV$. Here $P \leq H$ and $[P, g] \leq S$. Since $g \in XY$ we obtain

$$[\overline{P}, \overline{g}] = [\overline{P}, \overline{x}][\overline{P}, \overline{y}] \leq \overline{XY},$$

and since \overline{XY} is a p' -group we get $[\overline{P}, \overline{g}] = 1$. As $\overline{X} \cap \overline{Y} = 1$ we have $C_{\overline{XY}}(\overline{P}) = C_{\overline{X}}(\overline{P}) \times C_{\overline{Y}}(\overline{P})$. As $\overline{g} = \overline{xy}$ it follows that \overline{x} and \overline{y} centralize \overline{P} . Thus $P^x \leq (X \cap Y)P$ and $P \in \text{Syl}_p((X \cap Y)P)$. Thus there exists $z \in X \cap Y$ with $P^x = P^z$. Replacing (x, y) with (xz^{-1}, zy) we get $g = (xz^{-1})(zy)$ and $P \leq S_{(xz^{-1}, zy)}$. This contradicts the maximality of $\dim(Q)$ and (at long last) yields a contradiction which proves (a).

In order to prove (b): let $w = (g_1, \dots, g_n) \in \mathbf{W}(\mathcal{MN}) \cap \mathbf{D}$, and set $Q = S_w$. By 4.11 we may write $g_i = x_i y_i$ with $x_i \in \mathcal{M}$, $y_i \in \mathcal{N}$, and with $S_{g_i} = S_{(x_i, y_i)}$. Set $w' = (x_1, y_1, \dots, x_n, y_n)$. Then $w' \in \mathbf{D}$ via Q and $\Pi(w) = \Pi(w')$. Since each y_i normalizes U , it follows from 4.4 that $\Pi(w') = \Pi(w'')$ for some w'' such that $w'' = u \circ v \in \mathbf{D}$, where $u \in \mathbf{W}(\mathcal{M})$, and where $v \in \mathbf{W}(\mathcal{N})$. Thus \mathcal{MN} is closed under Π . In order to show that $\mathcal{MN} = (\mathcal{MN})^{-1}$ we note that if $(x, y) \in \mathbf{D} \cap (\mathcal{M} \times \mathcal{N})$ then $(y^{-1}, x^{-1}) \in \mathbf{D}$ and that $y^{-1}x^{-1} \in \mathcal{MN}$ by 4.2. Thus \mathcal{MN} is a partial subgroup of \mathcal{L} . Moreover, we have shown that $\mathcal{MN} = \mathcal{NM}$.

Let $g \in \mathcal{MN}$ and let $f \in \mathcal{L}$ with $(f^{-1}, g, f) \in \mathbf{D}$. As usual we may write $f = hr$ with $r \in \mathcal{N}$, $h \in N_{\mathcal{L}}(V)$, and $S_f = S_{(h, r)}$. Write $g = xy$ as in (a). By assumption we have $(f^{-1}, g, f) \in \mathbf{D}$ via some $P \in \Delta$. Setting $v = (r^{-1}, h^{-1}, x, y, h, r)$ it follows that $v \in \mathbf{D}$ via P and that $g^f = \Pi(v)$. Here $(h^{-1}, h, y, h) \in \mathbf{D}$ via $S_{(y, h)}$ by 4.2, so $v' := (r^{-1}, h^{-1}, x, h, h^{-1}, y, h, r) \in \mathbf{D}$ via P . Then

$$g^f = \Pi(v) = \Pi(v') = (x^h y^h)^r \in (\mathcal{MN})^r.$$

Since $r \in \mathcal{N}$, and \mathcal{MN} is a partial group, we conclude that $g^f \in \mathcal{MN}$. Thus $\mathcal{MN} \trianglelefteq \mathcal{L}$.

Set $M = N_{\mathcal{L}}(S)$, $N = N_{\mathcal{N}}(S)$, and let $s \in S \cap \mathcal{MN}$. Then (a) yields $s = fg$ with $f \in \mathcal{M}$, $g \in \mathcal{N}$, and with $S = S_{(f, g)}$. Thus $f \in M$ and $g \in N$, where M and N are normal subgroups of the group $N_{\mathcal{L}}(S)$. Then UV is a normal Sylow p -subgroup of MN ,

and since $s = fg \in MN$ we obtain $s \in UV$. Thus $S \cap \mathcal{MN} = UV$, and the proof of (b) is complete. \square

Recall from 6.1 the notation concerning \mathcal{K} , T , and $\overline{\mathcal{L}}$. Recall also the notion from 4.5, of an \uparrow -maximal element of \mathcal{L} with respect to \mathcal{K} .

Lemma 6.4. *Assume the setup of 6.1, and let $\overline{g} \in \overline{\mathcal{MN}}$. Then there exist elements $x \in \mathcal{M}$ of and $y \in \mathcal{N}$ satisfying the following conditions.*

- (a) $(x, y) \in \mathbf{D}$ and $\overline{g} = \overline{xy}$.
- (b) x , y , and xy are \uparrow -maximal with respect to \mathcal{K} .
- (c) $S_{xy} = S_{(x,y)}$.

Proof. By 5.7, $\overline{\mathcal{M}}$ and $\overline{\mathcal{N}}$ are partial normal subgroups of the locality $\overline{\mathcal{L}}$, and $\overline{\mathcal{M}} \cap \overline{\mathcal{N}} = \overline{\mathbf{1}}$. Then 6.3(a) yields a pair of elements $\overline{x} \in \overline{\mathcal{M}}$ and $\overline{y} \in \overline{\mathcal{N}}$ such that $(\overline{x}, \overline{y}) \in \overline{\mathbf{D}}$, $\overline{g} = \overline{xy}$, and $\overline{S}_{\overline{g}} = \overline{S}_{(\overline{x}, \overline{y})}$. Note that any preimage x of \overline{x} in \mathcal{L} lies in \mathcal{M} by 5.3(a) (and similarly for a preimage y of \overline{y}). A coset $\mathcal{K}f$ of \mathcal{K} is maximal if and only if f is \uparrow -maximal, by 4.13(c), so 5.3(a) implies that we may choose such preimages x and y so that x and y are \uparrow -maximal with respect to \mathcal{K} . Here x and y are in $N_{\mathcal{L}}(S \cap \mathcal{K})$ by 4.8, and $(x, y) \in \mathbf{D}$ by the definition of $\overline{\mathbf{D}}$ in 4.15. Thus (a) holds.

We now require:

- (1) $S_f \leq S_{(x,y)}$ for each $f \in \mathcal{L}$ such that $\overline{f} = \overline{g}$; and if $f \in N_{\mathcal{L}}(T)$ then $S_f = S_{(x,y)}$.

In order to prove (1) note first of all that since x and y are \uparrow -maximal, we have $\overline{S}_x = \overline{S}_{\overline{x}}$ and $\overline{S}_y = \overline{S}_{\overline{y}}$ by 5.3(c). Hence:

$$\overline{S}_{\overline{g}} = \overline{S}_{(\overline{x}, \overline{y})} = \{\overline{a} \in \overline{S}_{\overline{x}} \mid \overline{a}^{\overline{x}} \in \overline{S}_{\overline{y}}\} = \{\overline{a} \in \overline{S} \mid a \in S_x \text{ and } \overline{a}^{\overline{x}} \in \overline{S}_{\overline{y}}\}.$$

Here $a \in S_x$ for each $a \in S$ such that $\overline{a} \in \overline{S}_{\overline{x}}$, so

$$(2) \quad \overline{S}_{\overline{g}} = \{\overline{a} \in \overline{S} \mid a \in S_x \text{ and } a^x \in S_y\} = S_{(\overline{x}, \overline{y})}.$$

Now let $f \in \mathcal{L}$ such that $\overline{f} = \overline{g}$. As $T := S \cap \mathcal{K} \leq S_{(x,y)}$, (2) yields $TS_f = S_{(x,y)}$. In the case that $f T \leq S_f$ we get $S_f = S_{(x,y)}$, and this completes the proof of (1).

We now apply (1) to the element $f = xy$, and obtain $S_{xy} = S_{(x,y)}$. This completes the proof of (c), and it only remains to show that xy is \uparrow -maximal with respect to \mathcal{K} . Suppose that xy is not \uparrow -maximal. Then there exists $z \in \mathcal{K}$ such that $(z, xy) \in \mathbf{D}$ and such that S_{xy} is a proper subgroup of $S_{z(xy)}$. Then $T \leq S_{z(xy)}$. An application of (1) to $f = z(xy)$ then yields a contradiction, and completes the proof of (b). \square

Lemma 6.5. *Assume the setup of 6.1. Let $g \in \mathcal{L}$ with $\overline{g} \in \overline{\mathcal{MN}}$. Then $g \in \mathcal{MN}$, and there exist elements $x \in \mathcal{M}$ and $y \in \mathcal{N}$ with $(x, y) \in \mathbf{D}$, $xy = g$, and $S_{(x,y)} = S_g$.*

Proof. That g is in \mathcal{MN} is a consequence of 5.7. By 6.4 we may choose $x \in \mathcal{M}$ and $y \in \mathcal{N}$ with $(x, y) \in \mathbf{D}$, xy \uparrow -maximal with respect to \mathcal{K} , $\overline{g} = \overline{xy}$, and with $S_{xy} = S_{(x,y)}$. Then $\mathcal{K}xy$ is a maximal coset of \mathcal{K} by 4.13(c), and $g \in \mathcal{K}xy$ by 5.3(a). Thus there exists $z \in \mathcal{K}$ with $(z, xy) \in \mathbf{D}$ and such that $z(xy) = g$. As $S_{xy} = S_{(x,y)}$ we have also $S_{(z, xy)} = S_{(z, x, y)}$,

so that $(z, x, y) \in \mathbf{D}$ and $g = \Pi(z, x, y)$. Then $g = (zx)y$ by \mathbf{D} -associativity. As $\mathcal{K} \leq \mathcal{M}$ we get $zx \in \mathcal{M}$, and it now suffices to show that $S_{(zx,y)} = S_g$. As xy is \uparrow -maximal, the Splitting Lemma (4.12) shows that $S_g = S_{(z,xy)}$, and so:

$$S_g = S_{(z,xy)} \leq S_{(zx,y)}.$$

The reverse inclusion $S_{(zx,y)} \leq S_g$ is immediate from $g = (zx)y$. Thus, the lemma holds with zx in the role of x . \square

Given $g \in \mathcal{MN}$ and a pair of elements $(x, y) \in \mathcal{M} \times \mathcal{N}$ satisfying the conclusion of lemma 6.5, we shall say that (x, y) is an $(\mathcal{M}, \mathcal{N})$ -decomposition for g .

Theorem 6.7. *Let \mathcal{M} and \mathcal{N} be partial normal subgroups of \mathcal{L} . Then the following hold.*

- (a) $\mathcal{MN} = \mathcal{NM} \trianglelefteq \mathcal{L}$.
- (b) $S \cap \mathcal{MN} = (S \cap \mathcal{M})(S \cap \mathcal{N})$.
- (c) *There exists an $(\mathcal{M}, \mathcal{N})$ -decomposition for each $g \in \mathcal{MN}$.*

Proof. By 6.3(b), $\overline{\mathcal{MN}} = \overline{\mathcal{NM}}$ is a partial normal subgroup of $\overline{\mathcal{L}}$, and then 5.7 yields (a). Point (c) is given by 6.5. Now let $g \in S \cap \mathcal{MN}$, and let (x, y) be a corresponding $(\mathcal{M}, \mathcal{N})$ -decomposition. As $S_g = S$ we then have $x, y \in N_{\mathcal{L}}(S)$. Set $G = N_{\mathcal{L}}(S)$, $M = \mathcal{M} \cap G$, and $N = \mathcal{N} \cap G$. Further, set $U = S \cap M$ and $V = S \cap N$. Then $S = O_p(H) \in \text{Syl}_p(H)$, and we have $x \in M$ and $y \in N$. Set $K = M \cap N$, set $\overline{G} = G/K$, and adopt the usual “bar-convention” (as in 6.1) for homomorphic images. Then \overline{MN} is a direct product of torsion groups. By 4.1(c), U and V are maximal p -subgroups of M and N . Then (L3) implies that \overline{U} and \overline{V} are Sylow p -subgroups in \overline{M} and \overline{N} , and hence $\overline{U} \times \overline{V}$ is a Sylow p -subgroup of $\overline{M} \times \overline{N}$. Also $S \cap MN$ is a maximal p -subgroup, and hence a Sylow p -subgroup, of MN . Then $\overline{S \cap MN} = \overline{UV}$. As $S \cap K \leq U \cap V$ we obtain $S \cap MN = UV$, completing the proof of (b). \square

Theorem 6.8. *Let \mathfrak{N} be a non-empty set of partial normal subgroups of the locality (\mathcal{L}, Δ, S) , and let \mathcal{M} be the partial subgroup of \mathcal{L} generated by the union of \mathfrak{N} . Then $\mathcal{M} \trianglelefteq \mathcal{L}$. Moreover, for any well-ordering \preccurlyeq of \mathfrak{N} , \mathcal{M} is the set of all $\Pi(w)$ such that $w = (g_1, \dots, g_n) \in \mathbf{D}$, where each g_i is in some $\mathcal{N}_i \in \mathfrak{N}$, and where $\mathcal{N}_1 \preccurlyeq \dots \preccurlyeq \mathcal{N}_n$.*

Proof. Fix the well-ordering \preccurlyeq , and let \mathfrak{H} be the set of all partial normal subgroups \mathcal{H} of \mathcal{L} such that, for some non-empty subset \mathfrak{N}_0 of \mathfrak{N} , \mathcal{H} is the set of all elements in \mathcal{L} of the form $\Pi(w)$, $w = (h_1, \dots, h_n) \in \mathbf{D}$, with $h_i \in \mathcal{N}_i \in \mathfrak{N}_0$, and where $\mathcal{N}_1 \preccurlyeq \dots \preccurlyeq \mathcal{N}_n$. Then $\mathfrak{N}_0 \subseteq \mathfrak{H}$. Regard \mathfrak{N}_0 as a poset via inclusion, let Λ be a totally ordered subset of this poset, and set $\mathcal{H} = \bigcup \Lambda$. All calculations with finite subsets of \mathcal{H} take place in some member of Λ , so $\mathcal{H} \trianglelefteq \mathcal{L}$, and thus $\mathcal{H} \in \mathfrak{H}$. By Zorn’s Lemma there then exists a maximal $\mathcal{M} \in \mathfrak{H}$. Now let $\mathcal{N} \in \mathfrak{N}$. Then $\mathcal{MN} \trianglelefteq \mathcal{L}$ by 6.7, so $\mathcal{N} \leq \mathcal{M}$ by maximality. Thus $\mathcal{M} = \langle \bigcup \mathfrak{N} \rangle$. \square

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